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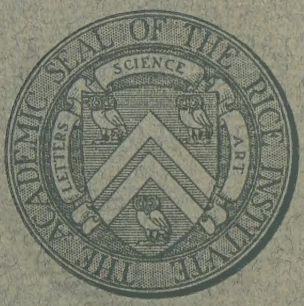

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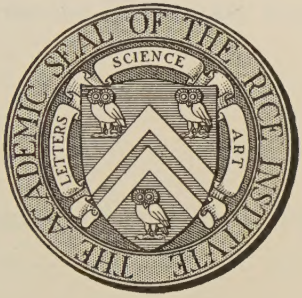

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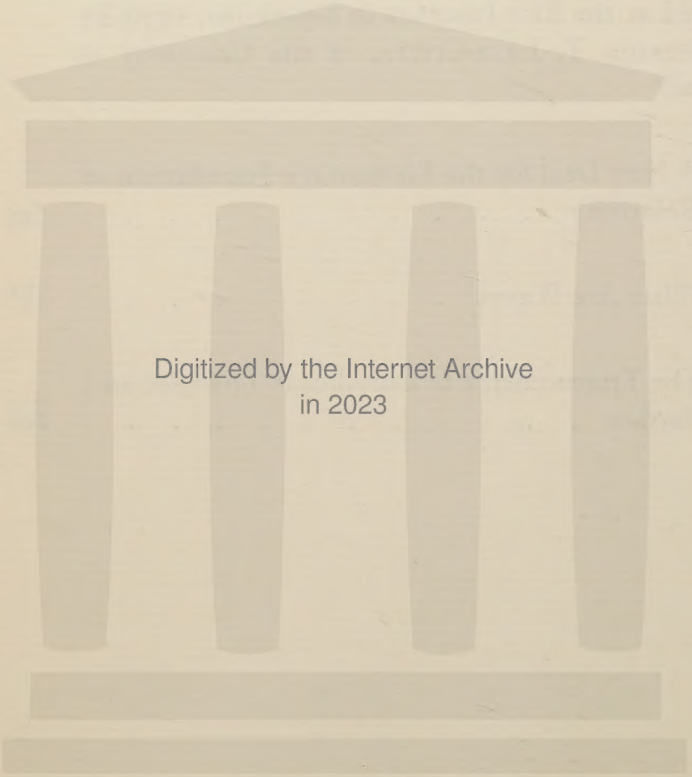

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I

A NEW DEAL FOR THE ELEMENTARY FOUNDATIONS OF RELATIVITY[1]

MY aim today is to present an elementary procedure for laying the foundations of relativity. This new approach will have the advantage of isolating the precise point where classical and relativistic conceptions diverge from each other.

Of course Einstein himself and his followers have not failed to fix, and even to embody in one or more mathematical statements, a certain set of assumptions, which are to be replaced by another relativistic set (extremely close to the former under ordinary conditions). The most popular and fascinating modification of the whole point of view is, beyond doubt, the one, first adopted by Einstein, which consists in a systematic replacement of ordinary translations by the corresponding Lorentz transformation. The formal background of this has now become quite familiar to anyone who has had, in school or from books, an introduction to this modern line of thought. But I shall avoid presupposing any previous knowledge of this argument and beg you to accept, as a historical hint, the statement that, in all the elementary or advanced expositions that I know, restricted relativity is built up by simply passing, somewhat abruptly, from one set of formulæ to another. Usually the writer contents himself with the remark that the new formulæ contain the old as a limiting case.

[1]This lecture and the two following ones were delivered at the Rice Institute in September, 1936, by Professor T. Levi-Civita, of the University of Rome.

I regard it, however, as worth while to give attention to a careful survey of the traditional assumptions used in establishing the first principles of kinematics, especially those having regard to time, meanwhile searching for the essential point where a certain classical idea and the corresponding intuitive postulate have to be abandoned and replaced by another, as intuitive as possible, but implying relativistic and not classical consequences.

This new deal for the elements of relativity claims to be something more than an axiomatic anatomy of its logical ingredients; for the following reason. For the purposes of physics, astronomy, and engineering, a thorough training in classical mechanics cannot be omitted, and, as statical and dynamical sensibilities are valuable, they must still be developed in students of natural philosophy and engineering. It is therefore a sound policy, not only from a methodological point of view but also from a pedagogical one, to connect relativity with the foundations of school training, emphasizing the common geometrical basis and the common reckoning and measuring of time for separate observers and thus fixing the attention on the exact moment in which the two lines of thought cease to be the same. This occurs when individual notions of time, acquired by *moving* observers, come into comparison. The traditional kinematics accepts the simpler of a certain pair of alternatives, thus establishing absolute time. The other alternative is far more general, and needs to be duly specialized in order to supply the relativistic scheme.

Now to proceed with a rapid development of this program.

I. THE ORDINARY NOTION OF TIME FOR A GIVEN OBSERVER

I take for granted that the familiar Euclidean geometry holds for the ordinary astronomical space S. The "fixed

stars," between which no appreciable changes of relative position were detected until recent times, used to offer a natural frame of reference, a fixed point being a point which preserved its position with respect to the "fixed stars." Now we may conceive of replacing the old Newtonian frame by a statistical one (in a well defined sense) and thus of being allowed to attribute a concrete meaning to *fixed* objects or *observers* in cosmic space S. With this understanding I proceed as if I had to explain in a classroom the elements of ordinary kinematics. First of all comes the notion of time and the measure of time. Without speculating on its philosophical meaning, there is undoubtedly agreement in ascribing a concrete (I do not say the only concrete) origin of time to our physiological or psychological sensations. To avoid vagueness, let us first fix our attention on a unique observer O.

Successions of sensations, in particular those arising from rhythmic phenomena, like the pulse, the alternation of day and night, of seasons, and so forth, afford the first rough estimates of time intervals. Ancient tools, such as the clepsydra, or more complicated machines, essentially based on vibrations (free or properly adapted), lead to more refined evaluations, and finally to a time-scale, like that of a good clock, corresponding to a continuous parameter t. Thus we get the time t of O.

2. MANY OBSERVERS AT REST IN THE ORDINARY ASTRONOMICAL SPACE S

If there are many observers O, O', $\cdots$, each develops in the above manner his own time, whence arise as many variables t, t', $\cdots$, which are *a priori* independent of each other. However, there is no possible ambiguity if, in accordance with our most deeply-rooted intuition, in establishing the

correlations among $t, t', \cdots$, we suppose that $O, O', \cdots$, are all immersed in the physical space (an ordinary Euclidean space S), and are at rest in it: at rest, of course, with respect to some frame of reference having a definite meaning, e.g., in the old style, the fixed stars, or in more modern style, their statistical substitute.

Assuming that there is perfect homogeneity in the behavior of local phenomena about the various observers $O, O', \cdots$, we must admit that the phenomena from which the times $t, t', \cdots$, have been inferred go on in the same way. This implies that the instruments (clocks), by means of which $O, O', \cdots$, deduce their time-measures, have the same behavior at all these places; therefore intervals $\Delta t, \Delta t', \cdots$, corresponding to the duration of analogous phenomena (that is, identical except for position) are to be considered equal. Thus, if we choose any special arrangement to characterize the unit of time for O, the same arrangement will fix the unit of time for O', and we shall have for any other corresponding duration $\Delta t = \Delta t'$: t and t' will accordingly differ only by a constant T, which may of course depend on O and O'.

It remains to be seen whether, and why (by convenient devices) T may be determined. Some further obvious assumptions are needed concerning the exchange of signals between two observers.

3. LIGHT AND ITS PROPAGATION

We assume as a matter of fact the mere circumstance that a flash of light may be sent from one place to another; its propagation through the Euclidean space S under consideration, being rectilinear with a constant speed c. If our two observers O and O', whom we have supposed to be at rest in S, are separated by a distance l, a light flash will

take a time l/c to travel in either sense between O and O'. By employing such signals, it is seen at once how to correlate t and t'. Suppose that a signal starts from O at the instant t_0, of the time t, and arrives at O' at the instant t_1' of the time t'. Owing to the identity of durations, between any t and the corresponding t' there is a constant difference:

$$t' = t + T.$$

Therefore the arrival at O' takes place for O at the instant t_1 defined by

$$t_1' - T,$$

and the interval of time required by the light to travel is

$$t_1' - T - t_0.$$

Equating this to l/c we have the desired expression for T by means of experimental data (t_1', t_0, l and c). Of course the compatibility of the assumptions requires that for any choice of t_0, we get always the same T. This being granted, we may obviously reduce T to zero by a simple displacement of the origin of time of one of the observers, or of both in obvious correlation.

We may think of such a regulation of clocks as being performed at every point (assuming that the observer is at rest at this point) of the space S; and so we pass from the individual times t, t', $\cdots$ to a unique time t, which is not yet the absolute time of the traditional kinematics, but has many of its characteristics: just those (as we shall ascertain) which subsist also on a more general footing, especially from the relativistic point of view. If we wish to give a name to this kind of time, we may call it *pantopic*, that is, valid for any place in the manner just described.

The absolute time of the old conception will now be introduced and illustrated. Thus we shall grasp the crucial moment at which relativity departs from the classical path.

4. ABSOLUTE TIME

Let O designate as before an observer at rest in S, t being his time (which may be adopted for every other fixed observer). We may describe the motion of any other point P, assigning at every instant the position of P with respect to O, more precisely with respect to a Cartesian system $Oxyz$. According to the first elements of analytic geometry this amounts to regarding the coördinates x, y, z of P as functions of t. Now, if O' with coördinates φ, ψ, χ, is another point of S (possibly moving with respect to $Oxyz$), we may introduce a trihedron $O'x'y'z'$ having axes $O'x'$, $O'y'$, $O'z'$ parallel to Ox, Oy, Oz respectively. If x', y', z' are the coordinates of P with respect to this second frame of reference we obviously have the ordinary formulæ of transformation

$$x = x' + \varphi, \quad y = y' + \psi, \quad z = z' + \chi. \tag{1}$$

When O', like O, was an invariable point of S, we recognized, as a result of almost irrefutable assumptions, that its time t' is, or may be reduced so as to be, equal to t. Note that, among the circumstances justifying the relation $\Delta t' = \Delta t$, we essentially employed the homogeneity of space and of the behavior of phenomena around the two observers. Obviously such identity of behavior is no longer necessarily experienced if, O being at rest, O' is moving.

The classical kinematics completes its temporal postulates by assuming that, in establishing the time t' of O', as described, the possible motion of O' has no influence: therefore a unique variable t may be assumed, which plays the rôle of time for O' as well as for O. This is the exact significance of introducing the absolute time t, that is, a time valid for any observer, whatever his motion may be.

5. GALILEAN PRINCIPLE OF COMPOSITION OF VELOCITIES

From the preceding, the mathematical deduction of the Galilean composition of velocities is immediate—as given in elementary textbooks. One has only to differentiate the formulæ (1) with respect to t, where, in the more general case, all may depend upon t. We get accordingly

$$(2) \qquad \frac{dx}{dt}=\frac{dx'}{dt}+\frac{d\varphi}{dt},\quad \frac{dy}{dt}=\frac{dy'}{dt}+\frac{d\psi}{dt},\quad \frac{dz}{dt}=\frac{dz'}{dt}+\frac{d\chi}{dt}.$$

By definition, the absolute velocity of P with respect to O (which means with respect to the frame $Oxyz$) is nothing but the vector v having components dx/dt, dy/dt, dz/dt; and the analogous velocity (*frame-velocity*) of O' is the vector w with components $d\varphi/dt$, $d\psi/dt$, $d\chi/dt$. If we introduce furthermore the relative velocity v' of the same P with respect to O' (i.e., with respect to the second trihedron $O'x'y'z'$), namely the vector defined by the components dx'/dt, dy'/dt, dz'/dt (the absolute time t serving for O' as well as for O, in the classical conception), we may condense the formulæ (2) in the unique vectorial relation

$$(3) \qquad v=v'+w,$$

which states the rule of composition of velocities and is expressed in words as follows:

absolute velocity = relative velocity + frame-velocity;

a result deeply rooted in the traditional intuition of ordinary motions, e.g., when we compare velocities of some kind as they are estimated by observers at rest or by those in motion.

6. APPLICATION OF THE PRECEDING RULE TO THE PROPAGATION OF LIGHT AT THE EARTH'S SURFACE—THE MICHELSON EXPERIMENT

If we agree to regard the propagation of optical signals through S (the absolute Euclidean space of classical mechanics and astronomy) as a particular rectilinear movement with constant vector-velocity $\boldsymbol{c}$, the Galilean principle of composition will still be available. In passing from an observer O, at rest with respect to the fixed stars—or, as we may say in the old terminology, with respect to ether—to another observer O', moving in any manner, we shall have from (3)

$$\boldsymbol{c} = \boldsymbol{c}' + \boldsymbol{w}, \tag{4}$$

$\boldsymbol{c}'$ being the light velocity as it appears to O': in particular O' may be identified with an observer on our earth. This is the case we shall consider. Owing to the additive term $\boldsymbol{w}$, $\boldsymbol{c}'$ turns out to be in general different from $\boldsymbol{c}$, and furthermore light is no longer propagated isotropically for the various directions issuing from O'. The difference is very small because, as firmly established by astronomical observations, the velocity $\boldsymbol{w}$ (properly the absolute value of the velocity) of the earth is roughly 30 km/sec, while c is 300,000, that is 10^4 times as great.

Considering in particular the case of a light-ray traveling in the direction of w, first in the same sense and second in the opposite one, we get from (4) the scalar relation

$$c = c' \pm w,$$

or

$$c' = c \mp w.$$

This difference in the velocity of propagation, though relatively very small, should produce, in a famous experiment

conceived and executed for the first time by Michelson in 1881, a displacement of diffraction fringes well within the limits of sensitiveness of optical instruments. As a matter of fact, not only the original experiment of Michelson but later ones, performed with every care by him and other physicists, have all given negative results, that is, almost complete absence of the expected displacement, at any rate an amount much less—only a few hundredths of that predicted by the above representation of the phenomenon.

Hence the conclusion, afforded by experimental evidence, that in reality the propagation of light behaves as if

$$c' = c$$

either rigorously or with a closer approximation than would be predicted by the classical rule (4) for the composition of velocities.

7. NECESSARY REJECTION OF SOME OF THE ASSUMPTIONS WHOSE SIMULTANEOUS ACCEPTANCE IS CONTRADICTED BY THE MICHELSON EXPERIMENT

For more than twenty years the negative result, which we have just considered, has puzzled theoretical physicists. Obviously one at least of the premises, on which the expected but not realized optical effect rested, must be wrong. But which of these premises, all apparently so spontaneous and so immediately suggested by the simplest intuition, and confirmed by daily experience? Since both geometrical and kinematical principles were supposed to be beyond discussion, it was thought at first that the weak point might be the rigorous identification of the propagation of light-signals with points (particles) moving with constant velocity c: in which case the Galilean rule of composition would no longer be a necessary consequence of the assumptions, and the

contradiction would disappear. This explanation, very reasonable from a logical and intuitive standpoint, was worked out on different lines by Stokes and Ritz, among others; but these theories are very complicated, and though restoring adherence to the facts as far as the Michelson experiment is concerned, they introduced discrepancies with respect to other phenomena and were never regarded as definitive.

A pragmatic explanation was found by Lorentz, as everyone knows, by means of the electro-magnetic theory of light. From this standpoint Maxwell's equations hold rigorously; on the contrary the uniform translation of rigid bodies is only an approximation (sufficient for ordinary phenomena) to be accompanied, for more refined purposes as in Michelson's experiment, by a certain very small contraction of body lengths in the direction of motion and, furthermore, by a similar contraction of time intervals, as calculated by co-moving observers.

Clearly this manner of presenting a law of motion, as originating (through *ad hoc* accommodation) from a familiar but only approximate law, did not seem satisfactory from a philosophical point of view. The last step, as is very well known, was accomplished by Einstein, who showed that the very meaning of the modified translation is nothing but the mathematical expression of his principle of relativity.

We wish, however, to obtain a deeper and more general insight into the very nature of time for a moving observer, without previously introducing rigid bodies and analyzing their translational movements. Of course, to get relativity itself, something more, especially suggested by it, must be postulated. This will be in fact a certain intrinsic character of events, which seems not to have been explicitly stated, though obviously involved in the theory.

8. ABANDONMENT OF ABSOLUTE TIME— GENERAL FORMULA FOR THE COMPOSITION OF VELOCITIES

O' being an observer, in general moving with respect to O, we still have the transformation formulæ (1) to relate the absolute and the relative motion of P, having the coördinates x, y, z referred to the first frame with origin O, and the coördinates x', y', z', referred to the second frame with origin in O'.

Suppose that, t being still the time of O, there is no absolute time, so that the moving point O' possesses a proper time t', which is not necessarily identical with t, nor differs from t by a constant. Ignoring for the moment the specific relations between t' and t, we may still consider, as before, the formulæ (1) of transformation of parallel axes from the origin O to O', interpreting them as connecting two aspects of the motion of the same point P, having the coördinates x, y, z, with respect to the origin O, and the coördinates x', y', z', with respect to the other origin O'.

In the first aspect the rôle of time is played by t, in the second by t'. Obviously we may differentiate the equation (1) with respect to t, noting that

$$\frac{dx'}{dt}=\frac{dx'}{dt'}\,\frac{dt'}{dt},$$

and similarly for y' and z'.

Writing for brevity

(5) $$\frac{dt'}{dt}=k,$$

we have

$$\frac{dx}{dt}=k\frac{dx'}{dt'}+\frac{d\varphi}{dt},\quad \frac{dy}{dt}=k\frac{dy'}{dt'}+\frac{d\psi}{dt},\quad \frac{dz}{dt}=k\frac{dz'}{dt'}+\frac{d\chi}{dt};$$

dx/dt, dy/dt, dz/dt are, as before, the components of the absolute velocity v; dx'/dt', dy'/dt', dz'/dt' (in which t' takes the place of t) are correspondingly the components of the relative velocity v', i.e., of the velocity referred to the parallel axes drawn through the point O', which is supposed to move in any manner whatever. The formulæ just written give the vector relation

$$v = kv' + w, \tag{6}$$

which obviously differs from (3) only by the scalar factor k, and holds without any hypothesis.

Admitting as usual absolute time we should have $dt' = dt$, and then, from (5), $k = 1$; (6) reduces thus to (3).

Our purpose is simply to recognize what determination of k characterizes relativity. The task will be very easy, since we (in this year of 1936) know beforehand where to go: the situation would of course be different had not Einstein previously discovered relativity! On the way, however, we have ascertained that one may establish a more general kinematics without introducing absolute time and that, even from this standpoint, there is still a very simple rule for the composition of velocities.

Now let us seek out a substitute for the usual postulate $k = 1$, and especially for the alternative adopted in relativity, one which is capable of explaining the failure of the Michelson experiment.

9. ELEMENTARY EVENTS

We shall give this name to events which, though really occupying a small space and a small interval of time, may be concentrated in a unique point P of the space S, and a unique instant t (of the pantopic time, common to all observers at rest in S).

From the mathematical aspect an elementary event E is merely a point of the four-dimensional manifold corresponding to the couple (P, t) consisting of a point P of S, defined by its three coördinates, and an instant t. We may cite a great many facts which, with obvious abstractions, reduce to elementary events. For instance, the closing or opening of an electric circuit at a well determined place; the lighting or extinguishing of a lamp; the firing of a gun; someone's death; a collision, and so on.

10. Amount of unlikeness between two elementary events E_1 and E_2

If two elementary events E_1 and E_2 both happen in the same place P, but at two different instants t_1 and t_2, it is natural (disregarding qualitative diversity) to adopt the difference

$$\Delta t = t_2 - t_1, \tag{7}$$

or better, its absolute value $|\Delta t|$, or, what amounts to the same, a proportional quantity, as an adequate measure of the deviation between one event and the other.

Passing to the general case of two events $E_1(P_1, t_1)$ and $E_2(P_2, t_2)$, which occur not only at different instants but also at different places, we may of course adopt several devices to estimate the distinction.

First we may disregard the diversity of position, and fix our attention on $|\Delta t|$ alone. This preponderance bestowed on time leads substantially to the traditional conception upon which the ordinary kinematics and, more generally, all classical physics, is founded. But it is equally permissible to pay attention also to local diversity. By mathematical analogies we should be led to apply Gauss's criterion, which is the following: for any two entities (e.g., geometrical

figures) F_1, F_2, of the same sort, characterized by the values of n parameters, say $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ respectively, an adequate numerical estimate of the total deviation is furnished by the quadratic form

$$\sum_1^n{}_i (x_i - y_i)^2,$$

or, possibly, by some other positive definite function of the arguments $x_i - y_i$.

Precisely on this model rests the introduction of the expression of constraint in analytical mechanics. The general idea is obviously suggested by the image of two geometric points, in ordinary space, which, unless coincident, appear always essentially distinct.

But there exist also entities of a different kind, endowed with one or more dominating characters, for which the equality of these principal features ensures a sufficient, or even striking equivalence, while the identity is only an irrelevant circumstance. A good example is furnished by a periodic function $f(x)$. Let us consider two values x_1, x_2 of the variable x, and suppose that it is only the corresponding values of the function, $f(x_1), f(x_2)$, which have an actual interest. The diversity between x_1 and x_2 then has to be estimated in terms of the values $f(x_1)$ and $f(x_2)$. Therefore, if x_1 and x_2 differ only by integral multiples of $\bar{\omega}$ ($\bar{\omega}$ being the period of the function f), their deviation must be regarded as zero, and in general it will depend only upon the remainder after dividing $x_2 - x_1$ by $\bar{\omega}$. It is not worth while to inquire now whether the absolute value of this remainder, or its square, or some other function, is the most convenient expression for the deviation. What alone matters is the realization that the deviation must be ap-

preciated in close connection with the specific nature of things.

II. PROVISIONAL INTERVENTION OF AN AUXILIARY OBSERVER. OPTICAL CONTEMPORANEITY. ELIMINATION OF SINGLE OBSERVERS

Suppose that an observer O becomes acquainted with two events E_1 and E_2, in general occurring at different places P_1, P_2 and different instants t_1, t_2, by the transmission of signals emitted from P_1 and P_2 at the instants in which the corresponding events happened.

By the postulates of the propagation of light, recalled in §4, these signals will reach O at the instants t_1+r_1/c, t_2+r_2/c, where r_1, r_2, designate the distances OP_1 and OP_2.

If O does not belong to the straight line P_1P_2 the signals arrive at O from *different* directions. The same is true (the directions being opposite) if O lies on the straight line, *inside* the segment P_1P_2. This is enough to point out the diversity between the two events: for the moment we need not go further in searching for a quantitative representation.

If on the other hand O lies on the straight line P_1P_2 *outside* the segment, then light-signals, emitted towards O from the two stations P_1, P_2, reach O in the same direction and sense. The reception of both signals will occur at the same instant if, and only if,

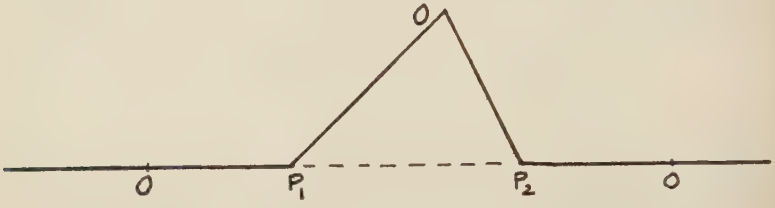
$$t_1+r_1/c=t_2+r_2/c.$$

In this case the two events appear to be contemporaneous to the observer. We shall then say that *they are synaesthetic for O*, that is perceived together by him.

Calling l the length of the segment P_1P_2 we have obviously

$$r_2-r_1=\pm l,$$

with the upper or the lower sign according as O lies (outside of the segment P_1P_2) beyond P_1 or beyond P_2.



The preceding *condition of synaesthesis, with respect to O*, may thus be written

$$\Delta t \pm l/c = 0, \tag{8}$$

where Δt already defined by (7), is merely the difference in times, $t_2 - t_1$.

Let us now consider the general case, in which, O lying outside of the segment P_1P_2, the condition (8) is not satisfied, the binomials

$$\Delta t + l/c, \ \Delta t - l/c,$$

being accordingly both different from zero. We shall try to derive from them some quantitative estimate of the deviation between E_1, E_2, *without intervention of the observer O*, who may henceforth be located anywhere, compatibly with his being able to receive at the same instant the signals of the two events, for which it is only required that O lies on the straight line P_1P_2 outside of the segment.

To this end we first multiply by c in order to get lengths and fix our attention on the absolute values of the two binomials

$$\begin{cases} \delta_1 = c\Delta t + l, \\ \delta_2 = c\Delta t - l. \end{cases} \tag{9}$$

Nobody will contest the statement that the more both

differ simultaneously from zero the larger the diversity between E_1 and E_2 is to be considered.

12. INTERVAL BETWEEN TWO EVENTS. EVENTS INFINITELY NEAR. ds^2 OF EINSTEIN-MINKOWSKI

Let us try to go further, introducing a criterion of measure. As already remarked, $\delta_1=0$, or $\delta_2=0$ represent, considered separately, the conditions under which the two events E_1, E_2 appear synaesthetic to observers located outside P_1P_2, either beyond P_1 or beyond P_2. Seeking now a definition of the diversity between the two events, in which both δ_1 and δ_2 are considered on an equal footing, we have to introduce some kind of mean: but what kind? Not the arithmetical one, whose absolute value

$$\tfrac{1}{2}|\delta_1+\delta_2| = c|\Delta t|$$

is merely (except for the constant factor c) the usual time difference, independent of the difference of position in space.

In seeking a mean, in which both elements are incorporated, the contemporaneity of optical perceptions being assumed as a test of equivalence (diversity then becoming zero), we are spontaneously led to think of the geometric mean, instead of the arithmetic; or, even more simply, the product

$$\delta_1\delta_2.$$

Indeed this quantity enjoys the double property of being symmetric with respect to δ_1 and δ_2 (vanishing when and only when either δ_1 or δ_2 does) and of increasing in absolute value with both $|\delta_1|$ and $|\delta_2|$.

All of this shows the plausibility of adopting as a measure of the diversity of two events E_1, E_2 the product $\delta_1\delta_2$, to be called the *world interval*. By (9), its numerical expression is

$$I = c^2\Delta t^2 - l^2. \tag{10}$$

In particular, if the two events $E_1(P, t)$ and $E_2(P', t+dt)$ are very near (in both time and space), writing dl for l, the interval I becomes the quadratic indefinite form ds^2 of Einstein-Minkowski:

$$ds^2 = c^2dt^2 - dl^2. \tag{11}$$

We have thus reached a genesis and a concrete interpretation of ds^2 which, though perhaps a little less familiar than some of those used in geometry and in mechanics, has still the advantage of implying only the simplest notions of geometry and of uniform rectilinear motion, without requiring, from the very beginning, the more complex analysis of rigid motions and therefore the motions of not only one point but of a system of points. This analysis may be deferred to a riper stage, the methodological foundation of relativity having been already established by a clear-cut, primary assumption. The above considerations lead us to regard intervals between two events as concrete physical entities, like distances and durations, which, however, are only particular cases.

The measures of these intervals are accordingly expressed by the formula (10), or even (11), if we are considering especially two events infinitely near.

Once having admitted the intrinsic character of the ds^2 corresponding to a given interval, it follows as a necessary consequence that this ds^2 is not affected by, i.e., is an invariant with respect to, our choice of any independent space-time variables. But a further tempting step is the attribution to it (as, for example, to force in classical mechanics) of intrinsic features: i.e., of existence and determination, completely independent of any observer, be he at rest or

moving at will through S. It is here that relativity begins to separate itself from traditional thought. According to the latter, it is admitted that the same variable t may represent time for any two observers O and O', irrespective of their rest or motion; the kinematics of relativity assumes, on the contrary, interval and not time as autonomous, i.e., independent of the observer.

It is hardly necessary to remark that, for the major part of terrestrial and astronomical events, we have to do with intervals for which ct is much larger than l, so that the invariance of interval is *approximately* the same as the invariance of Δt, so far as observers at rest in the S-space are concerned; but it is no longer the same for other more refined phenomena, especially optical ones: and it is these that called forth the theory of relativity and have provided very brilliant, although minute, experimental tests of the theory.

13. GENERAL CORRELATION OF TIMES

It is obviously impossible to go further in stating the immense influence of the preceding postulate on the development of all mechanics and physics. I must confine myself now to a very simple example, and to a general remark which makes clearer the essence of the Lorentz transformation.

First as an immediate consequence of the postulate, we may recognize what is (instead of $\Delta t = \Delta t'$) the relation between the times of two observers O and O', the former at rest in S, the latter moving in any manner.

The transit of O' through a position P at a certain instant t is obviously an event E; its transit at a successive instant $t+dt$, through another position P', infinitely near to P, is a second event E', infinitely near to E. The interval of these two events is expressed, according to (11), by

$$ds^2 = c^2dt^2 - dl^2,$$

where $dl = \overline{PP'}$. Since, actually, the ratio dl/dt is, by definition, the absolute value of the velocity of O' at the instant t, we may write

$$ds^2 = c^2(1 - v^2/c^2)dt^2. \tag{12}$$

Now let us apply the postulate affirming that the interval of any two events—in our case, the transits of O' through P at the instant t and through P' at the instant $t+dt$—may be estimated indifferently by the fixed observer O, giving the result (12), or by the moving observer O'. Adopting this plan, it will be remembered that the two events E and E' happen at the same place, namely at O' (passing first through P and then through P'). Therefore the ds in question is, except for the factor c, merely an elementary interval dt' of the time t', as it runs for O'. Accordingly

$$dt' = \frac{1}{c}ds = dt\sqrt{1 - v^2/c^2}, \tag{13}$$

and the coefficient k, which we have introduced generally to correlate the proper time t' of a moving observer O' with the standard time t, appears to be

$$k = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Obviously we must suppose $v < c$ in order to have real, finite, non-zero, values of k. In other words, the velocity c of light is to be regarded as a limit, never attained in the motion of real bodies.

For $v < c$ the relation (13) shows that, for a moving observer, the intervals dt' are smaller than the corresponding ones dt; that is, the clocks of O' go slower (contraction of times).

Permit me to finish with a striking, though almost trivial remark. It refers to the simplest form of motion of a body, kinematically conceived as a system of several points.

From the classical point of view, if we designate the actual positions (i.e., positions at a generic instant t) of these points by P, and their initial positions by P_0, we have clearly, for any t, a one-to-one correspondence between the P_0's and the P's. The same is true if we replace the initial positions P_0 by another manifold of points M in one-to-one correspondence with the P_0's, and therefore capable, like the P_0's, of individualizing the single moving particles. Accordingly, with self-evident abridged notation, we may write

$$P = P(M|t), \tag{14}$$

from which we get the vector-velocity v at any instant by partial differentiation with respect to t (M being regarded as constant):

$$v = \frac{\partial P(M|t)}{\partial t}. \tag{15}$$

From the classical point of view we have to consider t as absolute time, valid for any moving point. On the other hand, by the new relativistic postulate of the invariance of intervals, we have, for every moving point actually occupying the position P, a proper time t', whose elementary intervals dt' are shorter than the corresponding interval dt according to the formula (13), where v is the length of the vector (15).

Fixing the attention on a well determined point, M, as already stated, must be regarded as constant, and v is, by (15), a well determined vector function of t alone. Equation (13) may then be integrated, giving

$$t' = \int \frac{dt}{\sqrt{1 - v^2/c^2}} + \text{const.}$$

As the last term is intended to be "constant with respect to t," it may depend upon M, and the fourth relation above, to be associated with ordinary equations of motion (14), may be written

$$t' = \int_0^t \frac{dt}{\sqrt{1 - v^2/c^2}} + \chi(M), \tag{16}$$

χ being an arbitrary scalar function of M alone. The conclusion is as follows: while, in ordinary kinematics, the motion of a continuous system is defined by (14), t appearing in the right hand side as a parameter, the necessary association of (16) to represent the phenomenon of motion from a relativistic point of view, shows that, analytically, we have to do with a fourfold (instead of ternary) transformation between two quadruples (x, y, z, t) and (x', y', z', t'), x, y, z, indicating the coördinates (e.g., Cartesian) of M and x', y', z' those of P.

Among these general transformations, the simplest, from a relativistic point of view, are to be considered those for which not only the amount, but also the formal expression of the interval is preserved. They form a linear[1] ∞^9 named after Lorentz. It is precisely the group in four variables for which the quadratic form

$$ds^2 = c^2dt^2 - dl^2 = c^2dt^2 - (dx^2 + dy^2 + dz^2)$$

is left invariant, i.e., transformed (when expressed in terms of the new variables x', y', z', t') into the corresponding form

$$c^2dt'^2 - (dx'^2 + dy'^2 + dz'^2).$$

[1]If we add the qualitative condition that to finite values of x, y, z, t correspond finite values of x', y', z', t', and vice versa. Cf. a note by Dr. Clarice Munari in the *Rend. della R. Acc. dei Lincei*, ser. 5, XXIII, 1 Sem. 1914, pp. 781–787.

From this point on, everything goes on as it is set forth in good treatises on relativity. I trust, however, that I have contributed, by means of this preliminary introduction, a methodological improvement deserving of some didactic interest.

II

WHAT ARE WAVES?

IT is not easy to give a general definition of waves which would be precise and would at the same time include all cases presenting a character which our intuition attributes to waves. When we compare various kinds of phenomena which in some respect or other are connected with waves, and try to extract that which they have essentially in common, we perceive that certain circumstances, which at first sight seem characteristic, tend to efface themselves, so that one might be tempted to include (as Lagrange once did) all motion compatible with the laws of dynamics.

However, if we content ourselves with grasping an essential constituent (without pretending to include all cases which have some aspect suggestive of waves) we are led to adopt as typical the characterization enunciated in the fifteenth century by Leonardo da Vinci in the following form: "The impetus is much quicker than the water, for it often happens that the wave flees the place of its creation, while the water does not; like the waves made in a field of grain by the wind, where we see the waves running across the field while the grain remains in its place."

To render more precise this intuitive idea of Leonardo we will say that a motion of a material medium proceeds in waves, or that it is a progressive undulatory motion, when the actual displacements of the particles of the medium are accompanied by the much more accentuated and rapid motion of some striking constituent of the phenomenon in question: for example, the propagation in the medium of a

ertain perturbation (of which we shall recall, in §1, one of he best known aspects); or of some form of energy, or again to consider the typical case of waves in liquids) of a free urface, that is to say, the surface of separation between vater and air (§12), or finally, of a surface of separation etween two different regimes of motion of a material me- ium (§15).

With this understanding, I shall try to present to you a ummary of these different aspects, beginning, for the sake f greater clearness, with some quite classical generalities oncerning vibrating strings. I shall permit myself to dwell little on the elementary kinematic premises (§§1 to 8), in iew of the fact that they will permit me to bring in the otion of a group of waves (§9), and its consequences for ispersive media (§10), in a manner which is more direct nd illuminating than the usual one.

After making certain summarizing observations (§11) con- erning plane waves in other classes of phenomena (acoustic, lastic, electrical, optical) and especially concerning the ex- ression for the velocity of propagation as a function of the arameters which physically characterize the cases in ques- ion, I shall consider waves in the ordinary sense of the word, hat is, waves in liquids, giving an account especially of an- ient (§12) and modern (§13) conclusions bearing upon recti- near propagation in canals or in the open sea. It is thus hat we reach a point where, thanks to numerical compari- ons ingeniously conceived and carried out by M. H. Favre, rofessor at the École Polytechnique Fédérale of Zürich, it s possible to justify in a satisfactory manner the use of Gerstner waves, as they are now used in hydrography and n the theory of naval design, in spite of the serious incon- enience of requiring for their formation entirely artificial on-conservative forces. It is thus explained quite simply

how in ordinary circumstances it is possible, with as close approximation as may be desired, to replace the waves studied by myself in 1924, which fulfill exactly all physical requirements, with the much more elementary and tractable type discovered by von Gerstner about a century ago.

In §14 we limit ourselves to the mere mention of certain collateral questions; while §15 is devoted to waves of discontinuity, following the outline of Hugoniot and the very fruitful theory of Hadamard; and in §16 there is proposed a mathematical explanation of the dualism between waves and corpuscles which dominates modern physics.

I think it is well to draw attention to the fact that this lecture follows, in its essential lines, the article entitled "Onde—Teoria matematica dei fenomeni ondosi," published by Professor U. Amaldi and myself in the Enciclopedia Italiana.

I. VIBRATING STRINGS

In order to fix the ideas, we consider the phenomenon of transverse vibrations of a string. More precisely we examine the typical case of a segment of a string, homogeneous, flexible and sensibly inextensible, of length l, which is initially situated in coincidence with the rectilinear segment OA; and then by the intervention of suitable influences, vibrates in such a way that its various material elements perform small oscillations, in a well-determined plane passing through the line OA, and in a direction perpendicular to this line. Denoting by s the abscissa of a generic material element of the string in its initial rectilinear configuration, and by t the time, the transverse displacement which the element undergoes will be a certain function $\eta(s, t)$ of s and t; and from the theory of small oscillations of flexible inextensible strings (d'Alembert) it follows that the function

$\eta(s, t)$ must satisfy the second-order partial differential equation

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{T}{\rho} \frac{\partial^2 \eta}{\partial s^2}, \tag{1}$$

where the constants ρ and T represent respectively the linear density of the string (mass per unit of length) and the value of the tension in the initial situation. Besides satisfying this *indefinite* equation (which is to be valid at every instant and for every element of the string) the displacement must satisfy, as the case may be, depending on the connection of the string with other bodies, suitable boundary conditions (that is, relative to the end points O and A), which, in the particular case where the end points are fixed, reduce to the condition that at O and A the displacement is always zero, in other words, for *every* possible value of t, $\eta(0, t) = \eta(l, t) = 0$. Finally, from the very nature of the physical phenomenon, the maximum amplitude of the displacement η of each particle must be small in comparison with the length of the string; and this essential fact is introduced into the mathematical treatment of the problem by means of the hypothesis that the ratio η/l may be regarded as a quantity of the first order, that is to say, one of which the second and higher powers may be neglected. We take account of this hypothesis by supposing—as we may without contradicting the indefinite equation (1) and the more common types of boundary conditions—that the function contains an arbitrarily small constant factor ϵ, which fixes its order of magnitude.

In the sequel, in order to avoid developments which would not conform to the limitations of this lecture, we shall study exclusively the consequences which derive from the validity of the indefinite equation (1), admitting for consideration

once and for all only those solutions which satisfy the boundary conditions which, in each case, may be imposed by the problem treated. For this reason all that is to be said here will be valid unconditionally (that is, for every possible pair of values of s and t) in the case of an infinite string, for which no boundary conditions are present, but it will be valid only for intervals suitably delimited whenever it is necessary to take account of physical circumstances, representable in boundary conditions, as it would be in the case of a string, with reflection effects at the ends, and, more generally, with any phenomena of reflection or refraction in optics or any other field.

2. PERMANENT PROGRESSIVE WAVES

In order to render precise the wave aspects presented in the vibrations of the string, let us recall that the most general solution of the indefinite equation (1) is of the form

$$\eta(s, t) = \eta_1(s - Vt) + \eta_2(s + Vt), \tag{2}$$

where η_1 and η_2 denote two arbitrary functions of the respective arguments $s - Vt$, $s + Vt$ while V is a constant connected with the physical parameters ρ and T by the relation

$$V^2 = \frac{T}{\rho}; \tag{3}$$

and let us fix our attention on a solution (corresponding to the hypothesis that η_2 is identically zero) of the form

$$\eta = \eta_1(s - Vt), \tag{4}$$

making note of the fact that, in accord with the requirements of the preceding section, we must understand that the function η_1 contains as a factor an arbitrarily small constant ϵ. In this case the two variables s and t (element

of the string and the time) do not influence the phenomenon independently of each other, but only by way of the combination $s - Vt$, so that, if they vary in such a way that this binomial remains constant, then the corresponding displacement will remain unaltered. Now the condition

$$s - Vt = \text{const.}, \text{ or } s = Vt + \text{const.}$$

is none other than the equation of uniform motion along the line OA, with velocity V; whence we conclude that, during the vibration of the string governed by the equation (4), any observer, moving along OA with the constant velocity V, is confronted at every instant by a constant displacement η. In other words, it is as if the configuration assumed by the string at a given instant were displaced, without change, from O toward A with the velocity V. If then we take account of the fact that this velocity $V = \sqrt{T/\rho}$ is quite independent of the actual transverse velocity of every particle, given by $\partial\eta/\partial t$ and therefore of the same order of magnitude as η, namely, that of the factor ϵ, which is to be kept small, we realize that we are confronted with a typical wave phenomenon. To be precise, waves of this type are, for obvious reasons, said to be *permanently progressive;* the binomial $s_1 = s - Vt$ on which the phenomenon depends exclusively, is called the phase, and the velocity V, which is not connected with any material element, although dependent on the physical parameters ρ and T by means of (3), is called the *velocity of propagation of the wave* (and also, sometimes, the *velocity of phase*).

It is clear then that another solution of (1), which has, so to speak, the complementary form

$$\eta = \eta_2(s + Vt),$$

defines a permanent wave; and this wave differs from that

corresponding to the solution (4) only in the sense of propagation, inasmuch as the respective velocity is given, not by V, but by $-V$ (retrograde wave).

3. PERMANENTLY PROGRESSIVE WAVES IN MANY DIMENSIONS

In the case of phenomena in three dimensions also, whether acoustic, elastic, or electromagnetic, we come to the determination of waves, in seeking a particular class of solutions (of differential systems, or systems of partial differential equations, defining the phenomena in question) which depend on a single argument which is a linear function of the three space coördinates x_1, x_2, x_3 and of the time t, that is, of the type

$$\sigma = c_1x_1 + c_2x_2 + c_3x_3 + c_0\, t,$$

where the c's are constants to be suitably determined. Assuming that we are dealing with solutions which depend in some manner on the space coördinates (that is, are not functions of t alone) we must keep at least one of the quantities c_1, c_2, c_3 distinct from zero; that is, the vector $\mathbf{c}$ having c_1, c_2, c_3 as components along the coördinate axes must not be null. Thus we may regard as a single argument, on which the solutions in question depend, the expression $s_1 = \sigma/c$, where c denotes the length of the vector $\mathbf{c}$; and if we observe that the ratios $\alpha_i = c_i/c$ $(i = 1, 2, 3)$ are direction cosines (of the vector $\mathbf{c}$), and if we write $V = -c_0/c$, the expression s_1 thus takes the form $s_1 = s - Vt$, where $s = \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3$ is an abscissa in the direction $(\alpha_1, \alpha_2, \alpha_3)$. We are therefore dealing with *plane waves*, in the sense that the vibratory state depends, at every instant t, on s alone and is therefore identical at all points of a given plane $s = \text{const.}$; moreover, as in the case of the string, the phe-

nomenon has a stationary character with respect to any observer moving in the direction $(\alpha_1, \alpha_2, \alpha_3)$ so as to satisfy the condition $s_1 = s - Vt = \text{const.}$, that is, with the constant velocity V. A more general situation is obtained if we assume that s is any function (other than linear) of x_1, x_2, x_3, and suppose that the determining parameters of the phenomenon depend not only on $s_1 = s - Vt$ but also on another argument of purely spatial character. In this last type are included the so-called *spherical waves*.

But still more general types of waves, conceived in this manner, have been studied, from a different point of view, by H. Bateman (*Electrical and Optical Wave Motion*, Cambridge, 1915) and by G. A. Maggi (*Rend. Lincei*, ser. 5, XXIX, 1920, pp. 371–378).

4. BUNDLE OF WAVES

In the schematization of notable physical phenomena there arises the case of those permanent waves, for which the displacement η is (either rigorously or at least sensibly) different from zero only in a small neighborhood, of amplitude 2δ, of a certain value of the phase, e.g., $s_1 = 0$. From this it follows that at a generic instant t, the interval in which the vibration is sensible, lies between $s = Vt - \delta$ and $s = Vt + \delta$, so that its center is displaced in a straight line through the vibrating medium, with velocity V. It is this that is called, in one dimension, a *bundle of waves;* and it has its analogue also in two or three dimensions.

5. SINUSOIDAL OR HARMONIC PERMANENT WAVES

This important type of permanent waves is obtained if we suppose η to be of either of the two forms

$$\eta = \epsilon \sin\left(\frac{2\pi}{\lambda} s_1 + \alpha\right), \quad \eta = \epsilon \cos\left(\frac{2\pi}{\lambda} s_1 + \beta\right),$$

where λ, α, β denote constants, *a priori* undetermined. But these two expressions for η are reducible one to the other by putting $\beta=\alpha-\pi/2$; and on the other hand in either of them the constants α, β may be reduced to zero by changing the origin of space or time; hence it will suffice to consider the case

$$\eta=\epsilon\sin\left(\frac{2\pi}{\lambda}s_1\right)=\epsilon\sin\left(\frac{2\pi}{\lambda}[s-Vt]\right). \tag{5}$$

If we write, for brevity, $T=\lambda V^{-1}$, or

$$\lambda=TV, \tag{6}$$

and

$$x=2\pi\left(\frac{s}{\lambda}-\frac{t}{T}\right), \tag{7}$$

the equation (5) takes the simpler form

$$\eta=\epsilon\sin x. \tag{5'}$$

From the fact that $\sin x$ is a periodic function of x, of period 2π, it follows that (5) is a periodic function of the phase s_1, with period 1, and therefore periodic in s, likewise in t, with the periods T and λ, respectively. This means that, if we fix our attention on a fixed position (that is, if we assign a fixed value to s, e.g., $s=0$), we have for the transverse vibration of the corresponding particle a sinusoidal time equation which admits the period T, called the *period* of the wave motion in question (Fig. 1). If instead

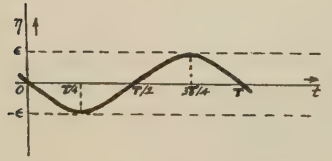


Fig. 1

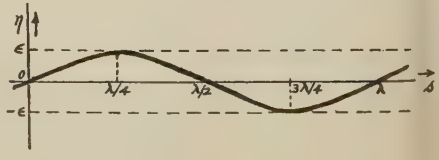


Fig. 2

we fix our attention on a given instant t (e.g., $t=0$), the law by which the transverse displacements of the various particles are governed in the vibrating medium, at this instant, is also sinusoidal, with period λ, which is called the *wave-length*, since the displacements are reproduced identically at intervals of length λ (Fig. 2). The two periods T and λ (which have the dimensions of time and length) are bound together by the relation (6), which represents the law of uniform motion with respect to the time. With these we consider also the respective reciprocals

$$\nu=\frac{1}{T},\quad k=\frac{1}{\lambda},$$

which are given the names *frequency* and *undulance*, and which enable us to give to the equation (5) of the sinusoidal wave the form

$$\eta=\epsilon\sin 2\pi(ks-\nu t). \tag{5''}$$

From the sinusoidal nature of the time equation of the transverse vibrations of a generic particle it follows that the corresponding displacement, in its periodic variations, reaches, at time intervals of length $T/2$, *maxima* with the value ϵ and *minima* with the value $-\epsilon$, which are designated by the common name wave-crests; and in the middle of the interval of time $T/2$, between the formation of one crest and the next, we encounter a *node*, that is an instant at which the displacement vanishes, and the particle passes through its natural position; thus $T/4$ is the time which is required for the particle to pass from a crest to the succeeding node, and vice versa. The progress of the diagram of displacements of the various particles at one and the same instant is analogous to the foregoing, with the sole difference that the distance (no longer temporal, but spatial)

from a crest to the succeeding node, or vice versa, is given by $\lambda/4$. Thus, in the typical case of a vibrating string, we can give a plastic model of the phenomenon by saying that: to an observer stationed at a point of the string, crests and nodes are formed alternately at intervals of time $T/4$; and, when he happens to see before him a crest or a node, he sees at the same instant formed at a distance $\lambda/4$ ahead of him and behind him a node O, or a crest, respectively.

6. PERIODIC PERMANENT WAVES IN GENERAL

The same qualitative considerations will also be valid obviously in the case where the displacement η, instead of being sinusoidal, is an arbitrary periodic function of period 1 of the single argument

$$\frac{2\pi}{\lambda} s_1 = 2\pi\left(\frac{s}{\lambda} - \frac{t}{T}\right) = 2\pi(ks - \nu t).$$

Particularly simple examples are obtained, if we return to the sinusoidal wave (5) and restore to its place in the argument the additive constant from which we freed ourselves at the beginning of the preceding section, that is, as we may say, by giving *phase* to a sinusoidal wave. We have thus

$$\eta = \epsilon \sin\left(\frac{2\pi}{\lambda}[s_1 - s_0]\right),$$

where s_0 denotes a constant having, like s_1, the dimensions of a length. We deal here with giving phase to typical sinusoidal solutions, and thus all it amounts to is a displacement by the amount s_0 of the space origin, which is equivalent to a shift of amount s_0 (in value and sign) of the diagram of Fig. 2. In particular, if s_0 corresponds to a forward or backward shift equal to one fourth of a period,

we pass from the diagram of the sine to that of the cosine or else to the negative of the cosine.

On the other hand we must call to mind that, by the theorem of Fourier, every periodic function $\eta(s_1)$ continuous and differentiable, or satisfying other conditions still less restrictive, may be represented as a sum, or, in the limit, as a series of sines and cosines of multiples of the independent variable, so that we obtain, if the period of $\eta(s_1)$ is 1,

$$\eta(s_1) = \frac{a_0}{2} + \sum_n \left[a_n \cos \frac{2\pi n}{\lambda} s_1 + b_n \sin \frac{2\pi n}{\lambda} s_1 \right],$$

or

$$\eta(s_1) = \frac{a_0}{2} + \sum_n c_n \sin \left(\frac{2\pi n}{\lambda} [s_1 - \sigma_n] \right),$$

where a_0, a_n, b_n or a_0, c_n, σ_n are suitable constants. In this way, if we set aside the constant term $a_0/2$ (which corresponds to a translation in the direction of the displacement, in general incompatible with the boundary conditions), the periodic wave turns out to be generated by the superposition of sinusoidal waves having, in order, wave lengths λ, $\lambda/2$, $\lambda/3$, $\lambda/4$, $\cdots$ and periods T, $T/2$, $T/3$, $T/4$, $\cdots$. All of this, naturally, is valid unconditionally in the case of an infinite vibrating string; when, on the other hand, we deal with a string of given length l, there intervene, as we have already noted, boundary conditions, which may be of various kinds and may influence the course of the phenomenon in an essential manner (giving rise, for example, to partial or total reflections at the ends, assumed to be fixed, or to particular dynamical conditions if an end is free). One who wishes, if only from an elementary viewpoint, to go further into these considerations, which must here be passed over, but which are particularly important sometimes in concrete

cases (elastic and, in particular, acoustical phenomena) may consult any of the treatises indicated in the bibliography.

7. PRINCIPLE OF SUPERPOSITION

When a phenomenon has to do with differential equations, or partial differential equations, which are linear and homogeneous, as in fact (1) is, the sum of two solutions is also a solution; and in this consists the so-called "principle of superposition," of which a particular case has been met with in the preceding section. But if we preserve a maximum of generality it is to be observed that, when we take account of all the characteristics of a given phenomenon, and, in particular, not only those of the indefinite equation, but also those of the boundary conditions, it happens, in general, that the sum of two solutions satisfying those conditions is no longer a solution of this kind, so that the principle of superposition must be applied with due caution. If, moreover, we consider, more particularly, problems relating to waves, it is necessary to note that, even in the favorable case in which it is true that, for the phenomenon in question, the sum of two solutions each corresponding to a propagation of waves is also a solution, it happens generally that the undulant character presented by each of these two solutions is no longer present in the sum; and a particularly striking example of this possibility is given by the general integral of (1)

$$\eta(s, t) = \eta_1(s - Vt) + \eta_2(s + Vt),$$

the terms of which represent two trains of waves, the first progressive, the second retrogressive, while the sum, by virtue of representing the most general vibratory motion of the string, does not possess in fact the character that distinguishes wave motion.

8. STANDING WAVES

A vibratory state, which deserves particular consideration, arises in correspondence with those solutions of (1) which are represented as products of two functions, one of s alone, the other of t alone,

$$\eta(s, t) = \tau(t)f(s).$$

In such a case the values of s which reduce $f(s)$ to zero also reduce η to zero, *whatever t may be*, so that the vibratory phenomenon has *fixed nodes*. Moreover, since we have

$$\frac{\partial \eta}{\partial s} = \tau(t)f'(s),$$

it follows clearly that those values of s, for which $f'(s)$ vanishes, also reduce $\partial\eta/\partial s$ to zero for *every* t. Thus the maxima or minima, if any, of the displacement η are *fixed*, and the crests and troughs likewise. A simple case, in which this situation is realized, is obtained when we superimpose two sinusoidal waves of equal amplitude, phase, length, and period, but one of them progressive and the other retrogressive; that is, by setting

$$\begin{aligned} \eta &= \epsilon[\sin 2\pi(ks - \nu t) + \sin 2\pi(ks + \nu t)] \\ &= 2\epsilon \cos 2\pi\nu t \sin 2\pi ks. \end{aligned} \tag{8}$$

This vibratory state is often referred to as a typical example of a "standing wave," but as a matter of fact the motion (8), resulting from the superposition of two trains of waves propagated in opposite directions, does *not* have the character of a wave motion according to the general criterion we indicated at the beginning.

9. GROUP OF PERIODIC WAVES AND GROUP VELOCITY

In the schematic case in one dimension (e.g., the vibrating string), any vibratory state is called, for obvious reasons, a "group of waves" if it is determined by the superposition of several permanent waves of the type considered in §2, and therefore corresponds to a displacement of the form

$$\eta = \eta_1 + \eta_2 + \cdots + \eta_n,$$

with

$$\eta_i(s, t) = \eta_i(k_i s - \nu_i t),$$

$i = 1, 2, \cdots, n$, where the η_i are all periodic functions, of period 1, of the respective arguments $s_i = k_i s - \nu_i t$; and, in general, the undulances k_i and frequencies ν_i vary with the index i. As we have noted in the case of a single component wave (§7) the phenomenon does not have in general a wave character; but if we fix our attention more particularly on the case $n = 2$, that is if we consider a binary group, which we will write, for simplicity,

$$\eta = \eta_1(ks - \nu t) + \eta_2(k's - \nu' t),$$

we can define a certain constant velocity U, the so-called *group velocity*, which is such that, with respect to an observer moving with this velocity in the direction of propagation, the phenomenon, without being properly undulant, turns out to be *periodic with respect to the time*. To show this, we consider first an observer Ω, moving with a constant generic velocity U; and, supposing that at the instant $t = 0$ he occupies the position $s = 0$, we denote by ξ the abscissa with respect to Ω of the generic particle of the vibrating system which has the absolute abscissa s. We have, for each instant, $s = \xi + Ut$, and, if in η_1, η_2 we let the rela-

tive abscissa ξ appear instead of s, the respective arguments become

$$k\xi-(\nu-kU)t,\quad k'\xi-(\nu'-k'U)t. \tag{9}$$

Since each of the two functions, η_1, η_2, is periodic, with period 1, with respect to its own argument, it is clear that if we select U so that the coefficients of t in the two arguments (9) are equal, the two functions will be periodic with respect to t, with one and the same period, equal to the reciprocal of the common value of the two coefficients of t. Thus we are led to set $\nu-kU=\nu'-k'U$; whence we obtain

$$U=\frac{\nu-\nu'}{k-k'}, \tag{10}$$

and this is the expression for the "group velocity."

This term seems quite justified if we consider the case in which both component waves are sinusoidal and the constants k', ν' are approximately equal to k, ν, respectively. In this case we have

$$\begin{aligned}\eta&=\epsilon\{\sin[2\pi(ks-\nu t)]+\sin[2\pi(k's-\nu' t)]\}\\&=2\epsilon\cos\{\pi[(k-k')s-(\nu-\nu')t]\}\sin\{\pi[(k+k')s-(\nu+\nu')t]\},\end{aligned}$$

or, introducing the relative abscissa ξ of the observer Ω, moving with velocity U, and taking account of the expression (10) for this velocity,

$$\eta=2\epsilon\cos[\pi(k-k')\xi]\sin\left[\pi\left\{(k+k')\xi+2\frac{k'\nu-k\nu'}{k-k'}t\right\}\right]. \tag{11}$$

In order to recognize the salient characteristic of this vibratory phenomenon with respect to the moving observer

Ω, it is to be noted that at every point whose relative abscissa is equal to

$$\xi_n = \frac{2n+1}{2(k-k')},$$

where n is any integer (positive or negative), we have rigorously $\eta = 0$, and in the neighborhood of any one of these points the displacement η is very small, so that we have what amounts to a region which is sensibly quiet (so-called "region of still water," if these generalities are applied to waves in canals, of which we shall speak a little later). These tranquil regions are separated by intervals of constant length $1/|k-k'|$, a quantity which, when k' is approximately equal to k, is much greater than either of the (practically equal) wave lengths $1/k$, $1/k'$ of the two component waves. In the interval extending between two consecutive regions of quiet the variation of η, which, inasmuch as it depends on two *distinct* arguments which are linear in ξ and t, is not undulant, gives place to a diagram which may wriggle in an arbitrary manner varying with respect to the time which would be referred to as "rippled," if we were dealing with the surface of a body of water. We see thus that the group velocity is that with which these successive wrinkled intervals, together with the intervening regions of tranquillity, are displaced with respect to a fixed observer.

10. DISPERSIVE PHENOMENA

In the preceding generalities, kinematic in character, we have not taken account of possible relations between wave-length and period, or, what amounts to the same thing, between undulance and frequency. When the two constants of one of these pairs are not independent of each other, so that one of them is a function of the other, the phenomenon

is said to be *dispersive*, inasmuch as, according to (6), the velocity of propagation V turns out to depend on the frequency $\nu=1/T$. In such cases the formula (10), giving the group velocity, acquires a particularly interesting significance. Taking as independent argument the undulance k, we can express as a function of this number, not only the frequency $\nu=1/T$, but also the velocity of propagation $V=\lambda/T=\nu/k$. Under the particular hypothesis of two progressive periodic trains of waves having approximately equal undulances and frequencies, we can take $k'-k$ as a differential dk and $\nu'-\nu$ as a differential $d\nu$ of the corresponding function, so that, by virtue of (10), we have for the group velocity the expression

$$U=\frac{d\nu}{dk},$$

and from this, remembering that $\nu=kV$, where V is the phase velocity—that is the velocity of propagation of the first wave train and, therefore, approximately that of the second also—we deduce

$$U=V+k\frac{dV}{dk}=V-\lambda\frac{dV}{d\lambda}.$$

When we deal with light phenomena, it is useful to introduce in the second term of this last expression for U, in place of V, the index of refraction $n=c/V$, where c denotes the velocity of light in empty space; and we arrive at the *formula of Rayleigh*

$$U=V\left(1+\frac{\lambda}{n}\frac{dn}{d\lambda}\right),$$

whence we deduce that, since, in general, the index of refraction decreases as the wave-length increases (normal dis-

persion)—so that we have $dn/d\lambda < 0$—in this case the group velocity is smaller than the phase velocity.

II. VELOCITY OF PROPAGATION IN CERTAIN IMPORTANT CASES

Analogously to what we have seen in the case of the transverse waves of a vibrating string (§2), as in every other type of physical phenomenon, when we seek those solutions of the corresponding differential equations which have the character of waves, we are led to express the velocity of propagation by means of the salient physical constants of the phenomenon. Thus, if first we turn to the case of a string, supposing it to be elastic, and consider its *longitudinal vibrations*, that is, no longer normal, but parallel to the direction of propagation, we find

$$V^2 = \frac{E}{\rho},$$

where ρ is as before the linear density, while E denotes the so-called *longitudinal modulus of elasticity of Young*, that is the ratio between a longitudinal pull, to which the cord is subjected, and the corresponding unit elongation.

For sound waves in air, considered as a perfect gas in an adiabatic system, we have

$$V^2 = \frac{\gamma p}{\rho},$$

where $\gamma = 1.41$ is the ratio of the two specific heats of air, corresponding to constant pressure and constant volume, and p, ρ represent the pressure and density of air when unperturbed by sound vibrations. As is well known, V in normal conditions turns out to be about 330 m./sec.

In the case of elastic waves in an isotropic medium we

find, denoting by λ and μ the so-called *elastic constants of Lamé*, either

$$V^2=\frac{\lambda+2\mu}{\rho}, \text{ or } V^2=\frac{\mu}{\rho},$$

according to whether we are dealing with longitudinal or transverse waves. Since the constants λ and μ are positive, we see that the velocity of the longitudinal wave is always greater than that of the transverse one; and it is worthy of note that both of these velocities turn out to be higher than those of sound in air. Inasmuch as λ and μ, while being in general markedly different, have on the other hand the same order of magnitude, the ratio of the first to the second velocity cannot be much inferior to 2; for example, in steel the two velocities are, respectively, 6100 and 3200 m./sec. It is perhaps worth while to add that, for isotropic bodies, Young's modulus is connected with the Lamé constants by the relation

$$E=\frac{(3\lambda+2\mu)\mu}{\lambda+\mu}.$$

In an elastic medium presenting a free surface, as happens, for example, in a semi-space (elastic stratum), waves may be propagated which are quickly reduced in intensity from the free surface toward the interior; these are called *superficial waves*. Considered in the first place by Rayleigh, they have been studied or generalized, with particular regard to seismic applications by L. De Marchi, C. Somigliana, R. Einaudi (in *Rend. Lincei*, ser. 5, XXV, 1916; XXVI, 1917, XXVII, 1918, ser. 6, XIX, 1934). The velocity of propagation of these waves is about equal to 95/100 of that of the transverse waves.

Finally, in a homogeneous isotropic dielectric, for which ϵ and μ are respectively the dielectric constant and the magnetic permeability, the velocity of propagation of electromagnetic and, in particular, of light waves is given by the celebrated *formula of Maxwell*

$$V^2 = \frac{c^2}{\epsilon\mu},$$

where c represents as usual the velocity of light in vacuo (to which, as we see, the above velocity for electromagnetic waves reduces when $\epsilon = \mu = 1$). If however the dielectric is anisotropic, while conserving magnetic isotropy, plane waves in all directions are still possible, but their velocities of propagation will vary from one direction to another. In so-called *biaxial media*, where we denote by V_1, V_2, V_3, the velocities of propagation of the waves normal to the optical axes, the velocity V, with which waves can be propagated normal to a generic direction having cosines α_1, α_2, α_3, is defined by the equation

$$\frac{\alpha_1^2}{V^2 - V_1^2} + \frac{\alpha_2^2}{V^2 - V_2^2} + \frac{\alpha_3^2}{V^2 - V_3^2} = 0,$$

and from this we obtain for the velocity a striking geometrical construction, by having recourse to the *Fresnel surfaces*, that is, those algebraic surfaces of order 4 (and of class 4), which admit with respect to these same axes the equation

$$\frac{V_1^2 x_1^2}{V_1^2 - r^2} + \frac{V_2^2 x_2^2}{V_2^2 - r^2} + \frac{V_2^2 x_3^2}{V_3^2 - r^2} = 1,$$

where $r^2 = x_1^2 + x_2^2 + x_3^2$.

12. WAVES IN LIQUIDS AND WAVES IN CANALS

We consider a rectilinear canal of rectangular section, with a horizontal bottom and vertical sides; and we suppose that

the liquid contained in the canal—which we shall speak of hereafter as water—moves parallel to the banks and has the same motion in all longitudinal sections, that is, in all vertical planes parallel to the banks. Lagrange, introducing for simplicity the hypothesis that the vertical accelerations of the separate particles of liquid are negligible in comparison with the acceleration of gravity g, considers *progressive waves of permanent type*, that is, those motions of the water in which the free surface l is displaced without alteration of form, progressing with a velocity V, while the separate particles, instead of being animated by a velocity comparable with V, execute only small oscillations with a local velocity having a mean value zero.

Now, in relation to the form of the free surface, the types most studied reduce to *periodic waves* (called by the French "la houle") and to *solitary waves*. It is quite clear that, in the first case the line l consists merely of consecutive equal arcs, repeated at constant intervals λ (wave-length); and the qualitative behavior of the curve is that of a sinusoid (Fig. 3). On the other hand, a solitary wave, such as might be produced by striking by suitable means, a mass of water in a canal initially at rest, consists of a single swelling,

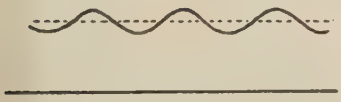


Fig. 3

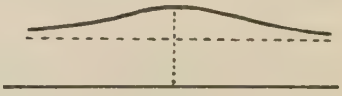


Fig. 4

symmetrical with respect to the maximum ordinate (Fig. 4). This was studied experimentally by Scott Russell (1844) and theoretically, in a wholly independent way, by J. Boussinesq (1871) and by Lord Rayleigh (1876).

A rigorously periodic type of waves is given by the so-called *trochoidal waves* discovered, independently, by F. J.

von Gerstner and W. J. M. Rankine. In these the curve of the free surface has exactly the form of a trochoid (or modified cycloid) and the separate fluid particles describe small circles which become rapidly smaller toward the bottom. The velocity of propagation of the waves is given in this case by

$$V^2 = \frac{g\lambda}{2\pi},$$

where, as usual, g denotes the gravitational acceleration and λ the wave-length. This solution of von Gerstner, by reason of the expressive simplicity of its characteristics, is widely used in hydraulic and nautical applications, as a theoretical basis for a first approach to the problem of estimating in some way the manifold perturbing influences which arise in practice; but it presents a certain inconvenience, whereby the application of the theoretical schema seems to be justified only provisionally, in the absence of something better (cf. §13); and this inconvenience consists in the *vortical* or *rotational* character of the corresponding vibrations of the liquid particles, whereas in a perfect liquid, under the action of conservative forces, it is possible to set up only *irrotational* motions.

It is therefore advantageous to seek other progressive waves of permanent type, which, while approximating as closely as possible to the simplicity of those of von Gerstner, will be due to irrotational vibrations of the liquid particles. Such indeed are the waves of Lagrange, to which we referred a little earlier, which have in fact particular importance in the study of tides; but the restrictive hypothesis to which their existence is subject (the negligibility in comparison with g, of the vertical accelerations of the fluid particles) is not in good accord with the physical facts. An

elementary type of periodic waves, permanent and irrotational, which lends itself better to applications, is that of the *simple sinusoidal waves* of G. B. Airy. Even these are approximate, but the approximation in this case corresponds to the physical nature of the phenomenon, in that it amounts to the assumption that it is possible to regard as a quantity of the first order the ratio of the maximum velocity of the particles to the velocity of the propagation V. The value of V is defined by *Airy's formula*

$$\frac{V^2}{gh} = \frac{\tanh \alpha}{\alpha},$$

where, letting λ and g have their usual meanings and denoting by h the depth of the canal, we have put

$$\alpha = \frac{2\pi h}{\lambda},$$

tanh α denoting the "hyperbolic tangent of α," that is, the quotient sinh α: cosh α. For the so-called *long waves*, that is, those having a length λ which is very large in comparison with the depth h (normally tidal waves are of this kind), α is small and tanh α/α may be regarded as unity, so that we obtain for the velocity V the expression

$$V^2 = gh,$$

already found by Lagrange in his approximation. With this same expression is to be reconciled that of the velocity of propagation of the single wave which is given by

$$V^2 = g(h+a),$$

where a denotes the height of the wave (height of the top of the swelling above the undisturbed level); and the analogy between the two expressions seems in a certain sense

justified, in that the single wave may be considered as a limiting case of a periodic wave, when the length λ becomes infinite. If, instead, we deal with *short waves* (and in practice we can regard as such all those for which the length is not greater than twice the depth of the canal) we can take $\tanh \alpha$ as unity, and there results

$$V^2 = \frac{gh}{\alpha} = \frac{g\lambda}{2\pi},$$

that is, we find again the expression which is valid for the velocity of propagation of trochoidal waves; and this result is not surprising, if we take account of the fact that the waves of von Gerstner, while being rotational, must in any case be considered as short, inasmuch as they have to do with a depth which is infinitely great.

For the simple waves of Airy, which, as we have said, correspond to a solution of first approximation, the principle of superposition (§7) is valid. In particular, we can consider the motion of the water which results from the superposition of two simple waves, having the same length, period, and amplitude, but directed in opposite senses and which gives rise, as we have seen in §8, to "standing waves," those having fixed nodes and crests. Of this type is the "clapotis" of the French.

In the case of waves in canals, besides the general property that the absolute motion of the liquid particles is small in amount compared with the velocity of propagation, the important mechanical fact is verified that in the deeper strata there is, in the mean, no global transfer of liquid: if the wave propagation does give rise to some transfer of matter, this is confined exclusively to the surface strata. Thus in the typical case of the simple waves of Airy, it is

as if there were, in the mean, a transfer of liquid with a velocity γ (very small compared with V) given by

$$\frac{\gamma}{V} = \frac{1}{2}\left(\frac{a}{h}\right)^2 \frac{gh}{V^2},$$

with the usual meanings for g, h, a; and from this it follows that the intensity of transfer varies directly as the square of the height of the wave and inversely as the velocity of propagation and the depth of the canal.

13. RIGOROUS SOLUTION OF THE PROBLEM

In the face of results as striking, though approximate, as these, it was natural to pose the mathematical problem of the *rigorous* determination of permanent and irrotational periodic waves. G. G. Stokes and Lord Rayleigh were prompted in this way to obtain further approximations; and the actual existence of an exact solution was proved, in the case of very deep canals, by T. Levi-Civita (1925), who gave also the formulæ suitable for actual calculation, and found for the velocity of propagation the expression

$$V^2 = \frac{g\lambda}{2\pi}\left(1 - \alpha^2 + \alpha^3 - \tfrac{3}{4}\alpha^4 + \tfrac{25}{12}\alpha^5 + \cdots\right),$$

where we have set

$$\alpha = \frac{2\pi a}{\lambda},$$

a being the maximum height of the wave above the mean level of the canal.

When a is negligible in comparison with the wave-length λ we are led back naturally to the short waves of Airy.

Following the researches of Levi-Civita, inquiry has been extended by D. J. Struik (1926) to the case of canals whose

depth is not extremely great, but well determined, while A. Weinstein (1926) has carried further the approximations of Boussinesq-Rayleigh for solitary waves. Particularly interesting on account of its theoretico-practical implications is a recent memoir by Professor H. Favre (*Publications of the Laboratory of Hydraulics of the Federal Polytechnic School of Zurich*, 1936), in which an accurate numerical comparison has been carried out between the exact solutions, namely, the irrotational ones (of Levi-Civita) and the vortical ones of von Gerstner, assuming equal values of λ (wave-length) and a (height) in the two cases. Favre has succeeded thus in establishing the fact that, for $a/\lambda \leq 1/20$, which is almost always the case for the types of waves which actually arise in practice, there is no sensible difference in the behavior of the lines of flow, nor in the other properties which can be tested experimentally.

This shows why the trochoidal waves of von Gerstner, justly appreciated for their geometrical simplicity and elegance, but used hitherto without sufficient justification (on the contrary, in spite of the explicit counter-indications arising from their vortical character), constitute an adequate representation, i.e., sufficiently approximate, of the physical phenomenon in the various cases, hydraulic, hydrographic, and nautical, in which it is customary to use them. This then is the complete explanation of a success, already achieved several decades ago, which, however, remained obscure, according to the standards of general mechanical principles, i.e., principles which are reasonably incontestable.

14. VARIOUS PROBLEMS OF WAVE MOTION

The limits of this lecture do not permit us to speak at length of other types of waves, however noteworthy, nor of their more complex characteristics, nor of the physical

problems of which these are the basis. We limit ourselves therefore to a simple enumeration: in one dimension, waves produced at the surface of separation of two liquids of different densities (J.-M. Burgers, N. Kotchine), and the influence on the production and propagation of waves exercised by capillary and by dissipative phenomena (friction, viscosity, turbulence, etc.); in two or three dimensions, waves induced by the motion of ships, small motions in very deep canals (so-called *waves of Poisson-Cauchy*), oscillations in locks, "seiches" in lakes, and, leaving the field of liquid motion, the whole theory of sound and wave phenomena on a large scale, oceanic and atmospheric, which are affected by the rotation of the earth.

15. WAVES OF DISCONTINUITY

In the present state of mechanics and physics one can say also that many natural phenomena (and, in the macroscopic field, almost all of them) find their mathematical representation in systems S of partial differential equations which involve a certain number of unknown functions φ_1, φ_2, $\cdots$, φ_m (characteristic physical parameters of the phenomenon), and in the case of three dimensions, four independent variables (three spatial coördinates x_1, x_2, x_3 and the time t); and these systems S are *normal*, that is, they contain as many equations as there are unknown functions and, as a result, they are solvable with respect to certain derivatives of maximum order. Each particular determination of the phenomenon, that is to say, every solution of the system S, remains uniquely individualized, when, after a three-dimensional variety (or hypersurface) Σ has been chosen arbitrarily in the four-dimensional space x_1, x_2, x_3, t, we preassign, for that matter arbitrarily, the functions to which the unknown φ_i and their derivatives are to reduce,

up to a certain order dependent on the nature of the system S. This is valid for a generic Σ; but we may have particular hypersurfaces Γ, called characteristics of the system S, for which it happens that the arbitrary elements just mentioned (functions to which the φ_i and their derivatives to a certain order reduce, on Γ) are no longer sufficient to determine a solution of S, inasmuch as there exist *infinitely many* solutions, all satisfying the preassigned conditions on Γ. If in such a case we fix, by means of further suitable conditions, two of these infinitely many solutions, e.g., φ_i and φ_i^*, mutually distinct, it results in general that these are defined on both sides of the hypersurface Γ; but if we limit ourselves to considering φ_i on one side and φ_i^* on the other, we have two solutions of the system S which are *in agreement* with each other only *partially* (that is, only to those orders reached by the arbitrary character of the initial elements on a generic surface Σ), while beyond those orders the two solutions present, with respect to each other, across Γ, certain discontinuous characters. Now, if we turn from the four-dimensional space x_1, x_2, x_3, t, to the ordinary space x_1, x_2, x_3, restoring to the fourth variable t its rôle as time parameter, the hypersurface gives place to a surface σ_t, variable with respect to the time; and the two solutions φ_i, φ_i^* define for the phenomenon in question two distinct regimes, valid, respectively, on opposite sides of σ_t, and this surface, instant by instant, marks a frontier of partial discontinuity between these two regimes. As time passes, this frontier σ_t is displaced and, in general, is deformed; and the consequent propagation of the discontinuity within the space (and in the continuous medium which may be involved in the phenomenon) presents the fundamental characteristics of a wave, which, in contrast with those considered hitherto, is called precisely a *wave of discontinuity*.

To make the motion precise, let a point P be fixed on σ_t relative to the instant t together with the corresponding normal, n, to the surface. When we pass to the instant $t+dt$, σ_t assumes a new configuration σ_{t+dt}, cutting n at a point Q close to P, which determines on n the direction of advancement of the wave and, with respect to σ_t (or, better, with respect to a sufficiently small region about P), a face, called the *wave front.*

This schematization, which goes back to Hugoniot and has been systematically developed by J. Hadamard, has made it possible to reduce to algebraic calculations, substantially, the study of the propagation of acoustic, elastic, electromagnetic, and other waves (but not those in liquids). It is not possible here to dwell upon the developments, for that matter relatively elementary, which permit us to deduce from a normal system S the definition of the *characteristic varieties* which it may have, nor upon the criteria, which give the procedure in the evolution of the wave surface from its initial configuration. A few observations will suffice.

Concerning the characteristic varieties of a normal system S, it is an important fact that in every case these are defined by a single partial differential equation (E), which is obtained from S in every instance by a well determined and valid process; but it may happen that it admits no real solution. For example, to Laplace's equation

$$\frac{\partial^2\varphi}{\partial x_1^2}+\frac{\partial^2\varphi}{\partial x_2^2}+\frac{\partial^2\varphi}{\partial x_3^2}+\frac{\partial^2\varphi}{\partial t^2}=0,$$

there corresponds as the equation (E) defining the respective characteristic varieties,

$$\left(\frac{\partial z}{\partial x_1}\right)^2+\left(\frac{\partial z}{\partial x_2}\right)^2+\left(\frac{\partial z}{\partial x_3}\right)^2+\left(\frac{\partial z}{\partial t}\right)^2=0,$$

which, in the real domain, is not satisfied by any function except $z=\text{const.}$; however, for the so-called *canonical equation of small motions* (of an incompressible fluid)

$$\frac{1}{V^2}\frac{\partial^2\varphi}{\partial t^2}-\frac{\partial^2\varphi}{\partial x_1^2}-\frac{\partial^2\varphi}{\partial x_2^2}-\frac{\partial^2\varphi}{\partial x_3^2}=0,$$

($V=\text{const.}$) the equation (E) is given by

$$\text{(E)}\qquad \frac{1}{V^2}\left(\frac{\partial z}{\partial t}\right)^2-\left(\frac{\partial z}{\partial x_1}\right)^2-\left(\frac{\partial z}{\partial x_2}\right)^2-\left(\frac{\partial z}{\partial x_3}\right)^2=0,$$

which admits real solutions possessing the maximum generality consistent with its order. Generally speaking, we may add that the equation (E) of the characteristic varieties certainly has real solutions (and therefore there exist waves of discontinuity) for all normal systems of the so-called *hyperbolic type*, while the contrary is the case for those of so-called *elliptic type;* and there is presented a whole series of intermediate cases of systems which are neither completely hyperbolic nor completely elliptic.

As for the various classes of phenomena in which the concept of waves of discontinuity has found fruitful applications, it will suffice to consider two cases of particular interest:

(i) If we are dealing with a phenomenon relating to a continuous *homogeneous* medium, it is proved that, under suitable specifications which express the physical fact that any other concomitant causes that there may be are independent of position, the equation (E), governing the evolution of the surface σ_t of the wave of discontinuity, is such that this surface, if initially plane, will remain plane at any other time; that is, homogeneous media admit in every case *plane waves of discontinuity.*

(ii) Turning to the case of an arbitrary phenomenon, we consider the possibility that the initial configuration σ_0 of σ_t is a very small *closed* surface, surrounding a given point P, and that across σ_t there are set up partially, in the manner sketched above, an interior solution different from zero and the solution identically zero in the exterior. We have here a representation of those phenomena which are initiated by a perturbation which is restricted to the immediate neighborhood of a given point, called the *epicenter;* in the course of time the wave front spreads and the perturbation is confined to the interior. When the physical characteristics of the phenomenon are constant, as happens in the propagation of sound or light in homogeneous media, the surfaces σ_t relative to successive instants are all homothetic with respect to the epicenter; and, in particular, if we are dealing with the propagation of light in a biaxial medium, the epicentral surfaces σ_t are homothetic to the surfaces of Fresnel (§11). Even more interesting is the case in which the perturbed region (non-zero solution) instead of penetrating the whole interior of σ_0 and hence also of σ_t, is always restricted to a small lamina contained between σ_0 and another characteristic surface interior to it and everywhere close to it; this corresponds to the ordinary case of a sound or light signal of short duration.

In every case a fundamental character of the phenomenon which completes quantitatively the notion of the wave front—corresponding to any element $d\sigma_t$ of the surface σ_t, relative to a generic instant—is the velocity, with respect to the trihedral of reference (for a fixed observer), with which the element of surface is displaced in the direction of its normal. This is the so-called *velocity V of advance* of the element of wave surface. If we are concerned with a specifically mechanical phenomenon, that is, involving along

with the motion of a surface of discontinuity, that of a material medium (air or some other gas or a solid body), there is to be considered besides the velocity V, the analogous velocity W, with which the element is displaced with respect to the medium. This is usually called the *velocity of propagation* and, by the principle of relative motion, is connected with V by the relation

$$V = W + V_n,$$

where V_n denotes the normal component (to the surface σ_t in the direction of propagation) of the velocity with which the material particle moves which is situated on the element $d\sigma_t$ at the instant t.

The distinction between the velocity of advance and that of propagation, which is essential for phenomena which are typically mechanical, has no *raison d'être* for phenomena which, like electromagnetic ones, do not imply the existence of a material medium; and in such cases we have usually a single velocity, which is strictly that of advance, but which, in the absence of any danger of ambiguity, is designated indifferently by the name velocity of propagation.

If we consider a given category of phenomena, with possible boundary conditions, on the one hand there may or may not exist progressive waves, and on the other hand (according to the type of the corresponding normal system) one may or may not have waves of discontinuity. Whenever waves of both kinds are possible at one time, it is found, at least in the cases studied hitherto, that the velocity of advance V of the wave of discontinuity admits the same expressions in terms of the physical parameters of the phenomenon which are valid for the corresponding progressive waves (§11).

But there are cases in which there exists only one of the two types of waves. For example, liquids are endowed solely with progressive waves (differential systems of elliptic type, or, at any rate, more nearly elliptic than hyperbolic); however, in homogeneous elastic media of general structure (depending on twenty-one structural constants) plane progressive waves are not possible (E. Beltrami), but there exist plane waves of discontinuity (G. Lampariello).

16. WAVES AND CORPUSCLES

The theory of waves of discontinuity, just referred to, opens the way for a remark of a general order on the dualism between the corpuscular and undulatory conceptions which, in the development of physics, after repeated vicissitudes in which one or the other has held the advantage, today tend toward agreement, as we said at the very outset, in a more advanced common point of view.

In the field of optics we usually attribute to Newton the corpuscular conception of light and to Huygens the undulatory conception, though in reality, from certain points of view at least, these are much older, and Newton himself, though giving preference to the emission theory, sometimes made use of undulatory models. As we have already pointed out, the great contest was decided when, following Young and Fresnel, it became possible to frame all phenomena known up to that time in an undulatory scheme, furnished first by an elastic model and later by the electromagnetic equations of Maxwell. But after all no one succeeded in reconciling in a simple manner the undulatory theory with the observed facts concerning the photoelectric phenomenon, which is due to H. Hertz; so that in order to represent these facts it became necessary to turn to the corpuscular point of view with the quantum hypothesis of A. Einstein,

according to which any beam of light of given frequency ν is to be regarded as made up of a swarm of photons (or quanta of light), small portions E of energy, each proportional to the respective frequency according to the formula $E=h\nu$, where h is the well known constant of Planck. Still later the same hypothesis made it possible to account for the Compton effect (1923), the complex nature of which turned out to be satisfactorily explained, as shown by Compton himself, P. Debye, E. Fermi, and E. Persico, when one adjoins to the hypothesis of Einstein not only the principle of the conservation of energy but also that of the conservation of the quantity of motion.

A corresponding change, but in the opposite direction, is presented in the theory of electrons. Whereas at the end of the nineteenth century, chiefly on the basis of the behavior of cathode rays and the celebrated experiments carried out mainly by J. J. Thomson, W. Kaufmann, H. A. Wilson, and R. A. Millikan, electrons had remained completely characterized as purely electric charges, all equal to each other, this point of view, exclusively corpuscular, was shown to be insufficient to account for the phenomenon of the diffraction of electrons in crystals, discovered in 1927 by G. Davisson and L. H. Germer, and confirmed by later experiments, due to E. Rupp and G. P. Thomson; hence here also it became necessary to have recourse to a complementary treatment of undulatory type.

This dualism, whereby the most notable facts of modern physics require the simultaneous intervention of corpuscles and waves, was recognized and regarded as a general law of nature, even before the beautiful confirmation of the diffraction of electrons by L. de Broglie, who sought from the first to give a more concrete form to his conception by associating with every moving corpuscle a well defined group

or bundle of waves. But he also recognized the difficulties involved in such an association.

Now, the considerations pointed out in the preceding section concerning waves of discontinuity are amenable to the formulation of a sufficiently broad mathematical schema to reconcile the two aspects, undulatory and corpuscular, of one and the same phenomenon, whenever it is adequately represented in a normal differential system. In fact, with the partial differential equation (E) of the first order which defines the wave surfaces, there is intrinsically associated (a classical result of Cauchy) a determinate ordinary differential system of equations which defines, in ordinary space, ∞^6 motions. Any one of these is represented in the space-time variety x_1, x_2, x_3, t by a line (time path), which constitutes a *bicharacteristic* of the equation (E) and, thereby, of the normal system governing the phenomenon. In this way, there are associated simultaneously with the phenomenon under consideration an undulatory aspect (wave of discontinuity) and a corpuscular aspect which seems to be schematized in the motions of real or fictitious particles along the bicharacteristics with the time law which is determined for each of them by the canonical system (C). There is, in short, for any physical theory (expressible in a normal differential system), the possibility of waves of discontinuity, resolvable mathematically into the motion of discrete particles.

This point of view is doubtless abstract in character and, from a physical point of view, agnostic; and, in particular it gives no criterion for a concrete linking of the two distinct kinds of phenomena. Manifold attempts in this direction carried out by eminent scientists, beginning with de Broglie himself, as we have said, have led to the conclusion that it is not possible to establish a one-to-one correspond-

ence between waves—or bundles of waves—and corpuscles, without violating the so-called principle of indetermination of Heisenberg. One can, however, broaden the question and seek to lay down a law of correspondence, not between individual elements but between trains of waves on the one hand and swarms of corpuscles on the other. In this way we succeed in establishing a correlation of a statistical type, on the basis of which, under suitable restrictions, we find and extend to more general cases the celebrated formula of de Broglie, which prescribes (in the absence of other perturbing phenomena) a well determined wave length λ to every type of electronic radiation, regarded from the corpuscular point of view. This is given by the formula

$$\lambda = \frac{h}{mv},$$

where h is Planck's constant and m and v denote, respectively, the material mass of the material particles constituting the radiation and their (mean) velocity.

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III

THE TRIGONOMETRY OF SMALL CURVILINEAR TRIANGLES ON A SURFACE

THE first notions concerning plane and spherical trigonometry date back to the Babylonians. The Greeks related these notions to Menelaos' theorem on transversals; substantial contributions have since been made by the Indians, by the Persians, and by the Arabs. One of these last, Nasir Ad-Din, prepared in the thirteenth century a treatise on pure trigonometry; that is, it did not deal directly with astronomy, though naturally it was influenced by the needs of that science.

The Renaissance saw the completion of the deduction of the relations, of which only three are independent, between the six elements, the angles and sides, of a triangle; at that time, these relations were given in the extremely diverse forms applicable to the various cases arising in practice. It is necessary to come on to Euler to find the formulæ of trigonometry in their actually compact form: previously, for want of having defined the trigonometric functions in such a systematic way that the definitions should hold for all real values of the angle, one was obliged to distinguish a multitude of cases and subcases.

An important generalization of trigonometry, from the theoretic point of view, was conceived by Gauss, who broached the study of geodesic triangles; that is, triangles formed by arcs of geodesics on an arbitrary surface. In this field appear researches of Christoffel, Weingarten, Darboux[1], and M. Severi.[2]

I propose now, as a further extension, the consideration not merely of geodesic triangles, but of triangles formed by arcs of arbitrary curves on an arbitrary surface. However, to obtain relatively simple results, we shall restrict ourselves to a consideration of *small* triangles.

But first of all, just what is a *small* triangle? If, in Euclidean geometry, we are considering rectilinear triangles, then smallness has no intrinsic sense, but depends on the unit of length; we can always, by a suitable choice, represent the lengths of the sides by numbers as small as we wish. The everyday criterion of smallness is determined here necessarily by additional considerations, depending on most varied circumstances. But, if the sides are curvilinear, there exists a natural yardstick: it is the *curvature*. To fix the ideas, let Γ denote the maximum of the curvature for the different points of the three sides, and let L denote the maximum length of the sides. Then the product ΓL is a *pure number*, and we are perfectly well justified in saying that a triangle is *small* if for it the above product is less than a certain proper fraction, which we can treat as a quantity of the first order.

Next, we are confronted with the problem of establishing, as in ordinary trigonometry, the relations between the lengths of the sides and the magnitudes of the angles of the triangle. In general, if the curves are arbitrary, we can attain only functional relations, that is, relations involving all features of the functions defining the curvilinear sides; but it is entirely different when we bear in mind that in taking recourse to the Taylor developments, we may utilize the smallness of the triangle to justify our considering as negligible, relative to unity, every quantity of order not less than 1, or 2, or 3, etc.

Continuing with our generalities, we hasten to remark

that we can very well consider the curvilinear triangle as being immersed directly in a containing Euclidean space, or more generally in a containing Riemannian space of an arbitrary number of dimensions; but it is convenient to consider first the case, which is undoubtedly the most interesting from the geodetic or topological point of view, where we suppose the triangle to be drawn on a given surface σ: a plane, sphere, ellipsoid, etc.

If we take account of the above surface σ, there is, except for the plane, another element, the total curvature K of the surface, which we must consider in order to appreciate the smallness of a triangle. Here again there is a pure number dependent on the dimensions of the triangle relative to those of the surface: it is the product κL^2, where κ denotes the maximum of K on the triangle and its interior. We make the hypothesis that if ΓL is a quantity of the first order, then κL^2 is of the second order; that is, is comparable with $(\Gamma L)^2$.

In this lecture, we shall discuss first (§1) small plane triangles whose sides are circular arcs, and shall outline in the most straightforward way the exact trigonometric relations, writing however explicitly only first approximation formulæ; that is, we shall neglect terms of the second and higher orders. This elementary introduction gives a first orientation to the analogous research in the general case of an arbitrary surface and of a triangle with arbitrary curvilinear sides, when we shall take account of terms of the second order but shall neglect those of order higher than the second. It is easy to see that, in order to pass to arbitrary curves and to our ulterior approximations, we must rely on a canonical representation of a curve in the neighborhood of a fixed point on the curve. We take the opportunity of reviewing the deduction of the representation

(§2), and of obtaining the extension to the case of curves on a given surface. This development demands quite a bit of preparatory material: intrinsic differentiation and formulæ of Frenet; expedient coördinates for a small region; and parametric expressions (§3) for the coördinates of a given curve on the surface. We study next triangles whose sides are geodesics (§4); in this study we find ourselves led, in the second approximation, to spherical trigonometry, either ordinary or Lobatchewskian (§5). We have now only to use the developments of the preceding sections to obtain, to the same order of approximation, the trigonometry of curvilinear triangles (§6).

I established the formulæ for the first approximation in 1934,[3] as a rather remote corollary of an extended geometric research concerning the mutual relations of three families of curves traced on a surface. More recently, the late Cohn-Vossen has shown[4] that we can obtain the same result by direct elementary considerations.

I had done this on my own account, without, however, publishing the remark, using the method which is presented here in §1, and which moreover is obtained from the same idea as that which inspired Cohn-Vossen. The passage to the second order, and eventually to higher orders (§8), follows, on the contrary, as we have indicated, from an analytic process admitting a systematic treatment.

I adjoin, for completeness, that §7 of this memoir contains a slight remark relative to the theorem of Legendre on the resolution of spherical triangles.

1. SMALL PLANE TRIANGLES FORMED BY THREE CIRCULAR ARCS

Let P_h, P_{h+1}, P_{h+2} be a plane triangle formed by three circular arcs (Fig. 1), with the obvious convention of re-

garding as identical those indices which differ by 3 or by a multiple of 3.

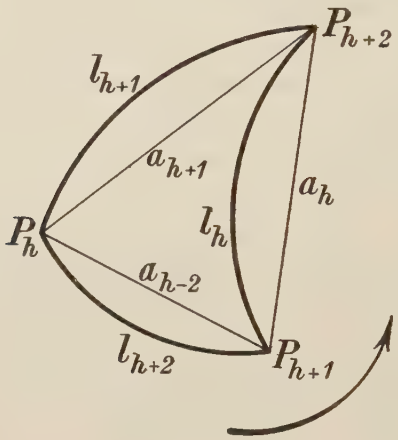

Fig. 1

We fix as positive sense of describing the boundary of the triangle, the sense which is defined by the increasing order of indices, designated by the arrow in the figure. And we represent by l_h the length of the side opposite the vertex P_h, that is to say, the length of the circular arc $P_{h+1}P_{h+2}$; by a_h the length of the corresponding chord $P_{h+1}P_{h+2}$. If R_h is the radius of the circle of which $P_{h+1}P_{h+2}$ is an arc, then the ratio l_h/R_h measures in radians the corresponding central angle. It goes without saying, that for *small* triangles, this angle is correspondingly small: in particular, we may always take this angle as being acute.

I introduce further the curvature γ_h taken with a *proper sign*, having $1/R_h$ for absolute value, and, for sign, $\pm$ according as the corresponding arc is situated on the exterior (as $P_{h+2}P_h$, P_hP_{h+1} in Fig. 1) or on the interior (as $P_{h+1}P_{h+2}$) of the rectilinear triangle $P_hP_{h+1}P_{h+2}$. The acute angle, at P_{h+1} or at P_{h+2}, comprised between the arc $P_{h+1}P_{h+2}$ and the

chord $P_{h+1}P_{h+2}$, is equal to half the central angle of magnitude l_h/R_h. It is convenient to give a sign to this angle by denoting it β_h, with

$$2\beta_h = l_h\gamma_h, \qquad h = 1, 2, 3. \tag{1.1}$$

The absolute value $|\beta_h|$ is thus precisely the acute angle between the arc and its chord; and, according to the definition of γ_h, the sign of β_h is $\pm$ according as the arc $P_{h+1}P_{h+2}$ is situated on the exterior or on the interior (as in Fig. 1) of the rectilinear triangle of chords.

It is easy to express the sides a_h and the angles α_h of the rectilinear triangle of chords, as functions of the elements l_h, φ_h of the circular triangle and of the curvatures γ_h. In putting these expressions in the elementary formulæ of rectilinear trigonometry, we shall obtain from them all the relations between the l_h, the φ_h, and the γ_h, which we propose to establish; these relations, duly simplified according to the smallness of the arcs l_h, relative to the respective radii, will be precisely the trigonometric relations for the circular triangles.

So much for the general thesis. Passing to the details, we first establish expressions for the angles φ_h in terms of the α_h and the auxiliary β_h.

If, as for P_h in Fig. 2, the circular arcs are concave toward the interior of the rectilinear triangle $P_hP_{h+1}P_{h+2}$, then β_{h+1} and β_{h+2} are both positive according to (1.1), and we have immediately

$$\varphi_h = \alpha_h + \beta_{h+1} + \beta_{h+2}. \tag{1.2}$$

This formula holds in every case. Indeed, if one of the circular sides abutting on P_h is, in the neighborhood of the point, interior to the rectilinear triangle, the corresponding

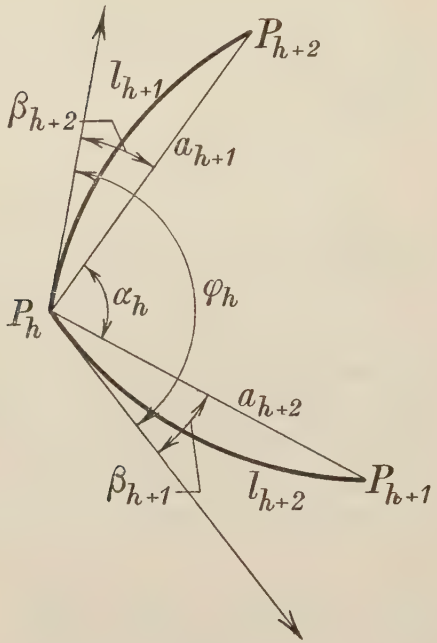

Fig. 2

β is negative by (1.1), and it is just this sign that we must give to the angle in the expression (1.2) for φ_h.

Since the chord a_h subtends the arc l_h, we have

$$a_h = 2R_h \sin(\tfrac{1}{2} l_h / R_h).$$

When we replace $1/R_h$ by $\pm\gamma_h$, the ambiguity in sign disappears in every case, and we retain

$$a_h = \frac{2}{\gamma_h} \sin(\tfrac{1}{2} l_h \gamma_h), \qquad h = 1, 2, 3. \tag{1.3}$$

Consider now any one of the formulæ of ordinary trigonometry. It is a relation between the a's and the α's, and can be rewritten by replacing a_h by its value (1.3), and α_h

by its value obtained from (1.2); that is to say, according to (1.1), by

$$\alpha_h = \varphi_h - \tfrac{1}{2}(l_{h+1}\gamma_{h+1} + l_{h+2}\gamma_{h+2}), \qquad h = 1, 2, 3. \tag{1.4}$$

If, and such is the case, we are content with first approximations with respect to the products $l_h\gamma_h$, we may neglect in the course of our calculations all terms of the second and higher orders relative to unity, and the results are very simple.

We give explicitly, by way of example, the formula for the sine function,

$$\frac{a_h}{\sin \alpha_h} = l, \qquad h = 1, 2, 3, \tag{1.5}$$

where l, independent of h, designates a small quantity of the order of the dimensions of the triangle; namely, l is the length of the diameter of the circumscribed triangle.

First of all, neglecting terms of the second and higher orders relative to unity, we can identify the sine with the arc, and we obtain as a consequence of (1.3) the result that we may write

$$a_h = l_h\{1 + ②\}, \tag{1.6}$$

where the symbols ①, ②, $\cdots$ designate a set of terms of order 1, 2, $\cdots$ at least, with respect to our argument $l_h\gamma_h$.

In order to obtain our trigonometric formula to the same order of approximation, we have only to replace in (1.5) the quantities a_h and α_h by their values (1.6) and (1.4).

Taking into account for the moment only the finite terms in α_h, we obtain from (1.6)

$$l_h = l\{\sin \varphi_h + ①\}. \tag{1.7}$$

Next, expanding $\sin \alpha_h$ according to (1.4), and taking into account only terms of the first order, we have

$$\sin \alpha_h = \sin \varphi_h - \tfrac{1}{2}(l_{h+1}\gamma_{h+1} + l_{h+2}\gamma_{h+2})\cos\varphi_h + ②.$$

In the terms of the first order we can replace l_{h+1} and l_{h+2} by their values as given in (1.7), and obtain

$$\begin{aligned}(1.8)\qquad \sin \alpha_h = \sin \varphi_h - \tfrac{1}{2} l(\gamma_h \sin \varphi_h + \gamma_{h+1} \sin \varphi_{h+1} \\ + \gamma_{h+2} \sin \varphi_{h+2})\cos \varphi_h + \tfrac{1}{2} l \gamma_h \sin \varphi_h \cos \varphi_h + ②.\end{aligned}$$

To abridge the notation, we introduce the triangular curvature τ; that is to say, the trinomial

$$(1.9)\qquad \tau = \tfrac{1}{3}(\gamma_h \sin \varphi_h + \gamma_{h+1} \sin \varphi_{h+1} + \gamma_{h+2} \sin \varphi_{h+2}).$$

We now obtain from (1.5), (1.6), and (1.8) the formulæ[5]

$$(1.10)\qquad \frac{l_h}{\sin \varphi_h} = l\{1 + \tfrac{1}{2} l(\gamma_h \cos \varphi_h - 3\tau \cot \varphi_h) + ②\}, h = 1,2,3,$$

which we desired to establish.

We see here a good example of the correction due to the curvature of the sides, which affects the relation of simple proportionality holding for rectilinear triangles. In (1.10) figures the auxiliary quantity l, which, according to (1.5), is the diameter of the circle circumscribed about the triangle of chords; that is to say, without mentioning the chords, the circle passing through the three vertices P_h.

In order to obtain a set of three independent relations between the sides l_h and the angles φ_h of the curvilinear triangle, it suffices to obtain an expression for l, as function of the l_h, φ_h, distinct from (1.10) itself. We obtain this immediately by taking the well known expression of l as function of the sides a_h of the rectilinear triangle, and then replacing each a_h by its value (1.3); this amounts substantially, according to (1.6), neglecting again here terms of the second order, to writing l_h in place of a_h.

For the triangle of chords, we have, for each h,

$$l=\frac{a_h}{\sin\alpha_h}=\frac{a_ha_{h+1}a_{h+2}}{a_{h+1}a_{h+2}\sin\alpha_h}.$$

The denominator is twice the area of the triangle; that is,

$$2\{p(p-a_1)(p-a_2)(p-a_3)\}^{\frac{1}{2}},$$

where

$$p=\tfrac{1}{2}(a_1+a_2+a_3) \tag{1.11}$$

designates the semi-perimeter.

Introducing the ratios

$$\frac{a_h}{p}=\mu_h, \qquad h=1, 2, 3,$$

which, since each side is less than the sum of the other two, are *proper fractions*, we can write

$$l=\tfrac{1}{2}p\frac{\mu_1\mu_2\mu_3}{\{(1-\mu_1)(1-\mu_2)(1-\mu_3)\}^{\frac{1}{2}}}. \tag{1.12}$$

According to (1.6), the definition (1.11) of p takes the form

$$p=\tfrac{1}{2}(l_1+l_2+l_3)\{1+②\}. \tag{1.13}$$

Thus, neglecting terms of the second order relative to unity, we see that p *is the semi-perimeter of the given curvilinear triangle;* similarly, according to (1.6),

$$\frac{a_h}{p}=\frac{l_h}{p}\{1+②\},$$

so that

$$\mu_h=\frac{l_h}{p}\{1+②\}. \tag{1.14}$$

In short, *the equation to associate with the theorem of sines* (1.10), *for plane curvilinear triangles, is* (1.12), *where* (as first approximation, the approximation adopted up to now) *we can calculate* p *and the* μ_h *from the given triangle:* p is the semi-perimeter and $\mu_h = l_h/p$, so that

$$\mu_1+\mu_2+\mu_3=2. \tag{1.15}$$

It is perhaps worth while to observe that, as a third equation to be associated with (1.10), where l has the meaning of an auxiliary, indeterminate constant, we may use

$$\alpha_1+\alpha_2+\alpha_3=\pi,$$

which is a goniometric identity for a rectilinear triangle, but becomes, according to (1.4), an effective relation between the elements of the curvilinear triangle under consideration.

In an analogous manner, we can generalize the other formulæ of ordinary rectilinear trigonometry, to the case of triangles formed by three circular arcs. I just point out this small old-fashioned task to those who wish to carry the calculations to the end.

2. LOCAL REPRESENTATION OF PLANE CURVES

Let a plane curve C be given, let Q be an arbitrary point on C, let s be the curvilinear abscissa of Q measured from a fixed point P of C, and let the curve be regular between P and Q inclusive. If x, y designate Cartesian coördinates, the functions

$$x=x(s),\ y=y(s), \tag{2.1}$$

which define the curve C in parametric form, shall be supposed to be continuous, together with all the derivatives which we shall have occasion to consider, along the arc PQ.

In the same sense, the point Q and the unit vectors t and n are regular functions of s, where t is directed along the tangent to C at Q, in the positive sense of s, and n is directed along the normal, in such a sense that the couple t, n is oriented as are the positive directions of the coördinate axes x, y.

For every point Q, we have the classical formula

$$\frac{dQ}{ds} = t, \tag{2.2}$$

and the formulæ of Frenet,

$$\frac{dt}{ds} = \gamma n, \tag{2.3}$$

$$\frac{dn}{ds} = -\gamma t, \tag{2.4}$$

where γ denotes the curvature of C *with sign*, and precisely with the sign $+$, if n is directed toward the concavity of C, but with the sign $-$ in the contrary case.[6]

Differentiating (2.3) with respect to s, and replacing dn/ds by its value (2.4), we obtain

$$\frac{d^2t}{ds^2} = -\gamma^2 t + \dot{\gamma} n, \tag{2.5}$$

where the superposed dot is an abridged notation for d/ds.

Applying the Maclaurin development to the point $Q(s)$, we have

$$Q(s) = P + st + \tfrac{1}{2}s^2\gamma n + \tfrac{1}{6}s^3(-\gamma^2 t + \dot{\gamma} n) + \cdots, \tag{2.6}$$

the terms not written being of order superior to the third with respect to s. Of course, the vector and scalar coefficients are all calculated at the point P.

Such a development can be extended by calculating the

successive derivatives of Q, and consequently of $\boldsymbol{t}$, according to (2.3), and by eliminating each time the derivatives of $\boldsymbol{n}$, according to (2.4) and its derivatives.

Formula (2.6) shows us that, in order to carry the development just to include the term in s^3, we must introduce γ and $\dot{\gamma}$. We see immediately, in reasoning by induction, that in carrying the development just to include the term in s^m, we must introduce the derivatives of γ at P through the order $m-2$.

We take account now of the essential circumstance that we are dealing with *small* arcs. Generalizing what we admitted in the introduction regarding the product ΓL, where Γ is the maximum of the curvature and L is the length of the small arc considered, we admit that the successive terms in the development of $Q(s)$ according to the powers of s, *which* (a factor s being taken separately) *are pure numbers*, have an order of smallness not less than $m-1$ for the term in s^m. If this is granted, we can in particular make formula (2.6) specific, as far as the order of approximation is concerned, by writing

$$(2.7) \qquad Q(s)=P+s\{\boldsymbol{t}+\tfrac{1}{2}s\gamma\boldsymbol{n}+\tfrac{1}{6}s^2(-\gamma^2\boldsymbol{t}+\dot{\gamma}\boldsymbol{n})+③\}.$$

If we suppose the origin of coördinates to be at the point P, we obtain a canonical representation of a plane curve in the neighborhood of one of its points.[7] We have only to take rectangular components in the above vectorial relation; that is to say, we have only to decompose the second member according to $\boldsymbol{t}$ and $\boldsymbol{n}$. We obtain

$$(2.8) \qquad \begin{cases} x=s\{1-\tfrac{1}{6}s^2\gamma^2+③\}, \\ y=s\{\tfrac{1}{2}s\gamma+\tfrac{1}{6}s^2\dot{\gamma}+③\}, \end{cases}$$

where ③ depends on the values of γ, $\dot{\gamma}$ on the entire arc PQ.

3. GEODESIC CURVATURE AND THE FORMULÆ OF FRENET FOR A CURVE DRAWN ON A SURFACE

On a surface σ we can consider, as in the plane, a curve C and a vector $\boldsymbol{w}$, which is a function of the variable point Q on C.

Let the surface be given in terms of any curvilinear coordinates x^1, x^2 whatsoever, let

$$ds^2 = \sum_{ik}^{2}{}_{1} a_{ik}\, dx^i dx^k, \qquad a_{ik} = a_{ik}(x^1, x^2), \tag{3.1}$$

be the expression of its linear element, and let w^i be the contravariant components of the vector $\boldsymbol{w}$.

We designate here too by s the length of arc on C, measured from a fixed point P, so that

$$x^1 = x^1(s), \quad x^2 = x^2(s) \tag{3.2}$$

are parametric equations of the curve C. Then the unit vector t, tangent to C at the variable point Q of curvilinear abscissa s, has for contravariant components

$$t^1 = \frac{dx^1}{ds}, \quad t^2 = \frac{dx^2}{ds}. \tag{3.3}$$

This being granted, it is necessary again for us to recall that we can generalize in an invariant manner the notion of the derivative of the vector $\boldsymbol{w}$ with respect to s, the length of the curve C, by defining this vector Dw by means of the components[8]

$$(Dw)^i = \frac{dw^i}{ds} + \sum_{jh}^{2}{}_{1} \Gamma^i_{jh}\, w^j \frac{dx^h}{ds}, \tag{3.4}$$

which, as easily could be verified, have contravariant be-

havior, where the Γ^i_{jh} denote the Christoffel symbols of the second kind for ds^2.

It follows furthermore from (3.4) that this same vector Dw has for covariant components

$$(3.5) \qquad (Dw)_i = \frac{dw_i}{ds} - \sum_1^2{}_{jh}\, \Gamma^j_{ih}\, w_j \frac{dx^h}{ds}.$$

We note in passing that, if w is defined *not only on* C, but also on a two-dimensional domain of σ, in the neighborhood of C, we can introduce the partial derivatives of w^i with respect to x^1 and with respect to x^2; we have then

$$\frac{dw^i}{ds} = \sum_1^2{}_h \frac{\partial w^i}{\partial x^h} \frac{dx^h}{ds},$$

and this permits us to write the contravariant components (3.4) of Dw in the form

$$(Dw)^i = \sum_1^2{}_h \frac{dx^h}{ds}\left\{\frac{\partial w^i}{\partial x^h} + \sum_1^2{}_j\, \Gamma^i_{jh}\, w^j\right\},$$

or again, since the quantity in brackets is the covariant derivative $w^i_{|h}$,

$$(Dw)^i = \sum_1^2{}_h\, w^i_{|h} \frac{dx^h}{ds};$$

from this last formula we perceive the contravariant character of the components $(Dw)^i$. For the case actually considered, of a vector w defined uniquely on C, I refer to my book previously cited.

If in particular we make the vector w coincide with t, we know that

$$Dt = \frac{1}{r} n,$$

where n denotes the surface vector normal to t toward the

concavity of C, and $\frac{1}{r}$ denotes the absolute value of the geodesic curvature. If $\boldsymbol{\nu}$ denotes $\pm n$, and we put for conformity

$$\gamma = \pm\frac{1}{r},$$

we obtain

$$Dt = \gamma\boldsymbol{\nu}, \tag{3.6}$$

where γ is the geodesic curvature of C, *with sign*, and precisely with the sign $+$ if $\boldsymbol{\nu}$ is directed toward the concavity of C, the sign $-$ in the contrary case. The vector $\boldsymbol{\nu}$ has the two properties of being a unit vector and of being perpendicular to t. In accordance with the metric of the surface σ, these facts are translated into the two formulæ

$$\sum_1^2{}_i \nu^i\nu_i = 1, \quad \sum_1^2{}_i \nu^i t_i = 0,$$

ν_i and t_i denoting, of course, covariant components.

Differentiating the first equation with respect to s, we obtain

$$\sum_1^2{}_i \left(\frac{d\nu^i}{ds}\,\nu_i + \frac{d\nu_i}{ds}\,\nu^i\right) = 0.$$

If we replace $d\nu^i/ds$, according to (3.4), by

$$(D\boldsymbol{\nu})^i - \sum_1^2{}_{jh} \Gamma^i_{jh}\,\nu^j \frac{dx^h}{ds},$$

and $d\nu_i/ds$, according to (3.5), by

$$(D\boldsymbol{\nu})_i + \sum_1^2{}_{jh} \Gamma^j_{ih}\,\nu_j \frac{dx^h}{ds},$$

we find after a reduction that

$$\sum_1^2{}_i [(D\boldsymbol{\nu})^i\nu_i + (D\boldsymbol{\nu})_i\nu^i] = 0.$$

The two sums are equal, each of them representing the scalar product of $D\boldsymbol{\nu}$ and $\boldsymbol{\nu}$, in accordance with the metric (3.1). This product is therefore null, whence we are assured that the vector $D\boldsymbol{\nu}$ is perpendicular to $\boldsymbol{\nu}$, on σ, and consequently has the same direction, if not the same sense, as $\boldsymbol{t}$.

We can therefore write

$$D\boldsymbol{\nu} = -\gamma^* \boldsymbol{t}, \tag{3.7}$$

γ^* representing a scalar factor as yet undetermined. In order to determine it, we have only to differentiate with respect to s the relation of orthogonality

$$\sum_1^2{}_i \nu^i t_i = 0,$$

and then to replace, as always, the derivatives $d\nu^i/ds$ and dt_i/ds by their values obtained from (3.4) and (3.5). We have thus

$$\sum_1^2{}_i [(D\boldsymbol{\nu})^i t_i + (D\boldsymbol{t})_i \nu^i] = 0.$$

But, in terms of covariant components, (3.6) is equivalent to

$$(D\boldsymbol{t})_i = \gamma \nu_i.$$

Similarly, for contravariant components, (3.7) gives

$$(D\boldsymbol{\nu})^i = -\gamma^* t^i.$$

Since $\boldsymbol{t}$ and $\boldsymbol{\nu}$ are both of length 1, the equality arising from differentiation becomes therefore

$$-\gamma^* + \gamma = 0.$$

The expression for $D\boldsymbol{\nu}$ acquires precisely the form

$$D\boldsymbol{\nu} = -\gamma \boldsymbol{t}, \tag{3.8}$$

which is exactly the second formula of Frenet for curves on σ, the first being (3.6).[9]

Returning to the contravariant components (3.4), putting $\boldsymbol{w}=\boldsymbol{t}$ and writing t^i in place of dx^i/ds, we obtain the two formulæ of Frenet (3.6) and (3.8) explicitly in the form

$$\frac{dt^i}{ds}=\gamma\nu^i-\sum_{jh}^{1,2}\Gamma^i_{jh}t^jt^h, \tag{3.9}$$

$$\frac{d\nu^i}{ds}=-\gamma t^i-\sum_{jh}^{1,2}\Gamma^i_{jh}\nu^jt^h, \qquad i=1,2. \tag{3.10}$$

For simplification, it is convenient to refer to isothermic coördinates x^1, x^2; that is, coördinates such that ds^2 reduces to the form

$$ds^2=\lambda(dx^{1^2}+dx^{2^2}). \tag{3.11}$$

It is convenient, moreover, to concern ourselves especially with the canonical form[10] relative to the point P, which is the origin of the arc s. Then, in a neighborhood of P, of either the first or the second order, everything behaves as if λ had the form

$$\frac{1}{1+\dfrac{K}{4}(x^{1^2}+x^{2^2})},$$

where the coördinates x^1, x^2 vanish at P, and where K denotes the total curvature of the surface σ at the point P. If we adopt the obvious notation

$$\lambda_i=\frac{\partial\lambda}{\partial x^i}, \quad \lambda_{ik}=\frac{\partial^2\lambda}{\partial x^i\partial x^k}, \qquad i,k=1,2,$$

we see at once that, *at the point* P,

$$\begin{aligned}&\lambda=1, \quad \lambda_1=\lambda_2=0;\\ &\lambda_{11}=\lambda_{22}=-K, \ \lambda_{12}=0.\end{aligned} \tag{3.12}$$

The values of the second derivatives can be written together in the formula

$$\lambda_{ik} = -\delta_{ik} K, \qquad i, k = 1, 2, \tag{3.13}$$

where the δ_{ik} are the symbols of Kronecker (0 for $i \neq k$, 1 for $i = k$).

We have, consequently,

$$\Gamma^i_{jh} = \frac{1}{2\lambda} \sum_1^2{}_l \, \delta_{il} \{ \delta_{lj}\lambda_h + \delta_{lh}\lambda_j - \delta_{jh}\lambda_l \}.$$

Since δ_{il} is zero for $l \neq i$ and equal to unity for $l = i$, we have simply

$$\Gamma^i_{jh} = \frac{1}{2\lambda} \{ \delta_{ij}\lambda_h + \delta_{ih}\lambda_j - \delta_{jh}\lambda_i \}. \tag{3.14}$$

Introducing these values in (3.9), replacing t^i in the first member by dx^i/ds, and using the relation

$$\lambda(t^{12} + t^{22}) = 1,$$

we obtain

$$\frac{d^2x^i}{ds^2} = \gamma\nu^i - \frac{t^i}{\lambda} \sum_1^2{}_h \lambda_h t^h + \frac{\lambda_i}{2\lambda^2}. \tag{3.15}$$

At the point P, we have simply

$$\left(\frac{d^2x^i}{ds^2}\right)_P = \gamma\nu^i, \qquad i = 1, 2. \tag{3.16}$$

Similarly, (3.10) reduces at the point P to

$$\left(\frac{d\nu^i}{ds}\right)_P = -\gamma t^i. \tag{3.17}$$

Differentiating (3.15) with respect to s, and taking account of the fact that, for a function $f(x^1, x^2)$ such as λ, λ_i, or λ_{ik},

$$\frac{df}{ds}=\sum_1^2{}_i \frac{\partial f}{\partial x^i}\ \frac{dx^i}{ds}=\sum_1^2{}_i \frac{\partial f}{\partial x^i}\,t^i,$$

we obtain from (3.16) the relation

$$\left(\frac{d^3x^i}{ds^3}\right)_P=\dot\gamma\nu^i-\gamma^2t^i-\frac{t^i}{\lambda}\sum_1^2{}_h\lambda_{hj}t^ht^j+\frac{1}{2\lambda^2}\sum_1^2{}_j\lambda_{ij}t^j,$$

where $\dot\gamma$ denotes $(d\gamma/ds)_P$, and where the omitted terms vanish at P.

Taking account of formulæ (3.13), we have

$$\left(\frac{d^3x^i}{ds^3}\right)_P=\dot\gamma\nu^i-\gamma^2t^i+Kt^i-\tfrac{1}{2}Kt^i=\dot\gamma\nu^i+(\tfrac{1}{2}K-\gamma^2)t^i. \tag{3.18}$$

At the point P, we have $\lambda=1$; and the components $t^1=dx^1/ds$, $t^2=dx^2/ds$ of an arbitrary direction $\boldsymbol{t}$ coincide, according to (3.11), with their direction cosines, that is to say, with $\cos\theta$, $\sin\theta$, where θ is the angle which the vector $\boldsymbol{t}$ makes with the positive direction of the x^1-line ($x^2=$ const.). Similarly, $\boldsymbol{\nu}$ has components

$$\nu^1=\cos\left(\theta+\frac{\pi}{2}\right)=-\sin\theta,\quad \nu^2=\sin\left(\theta+\frac{\pi}{2}\right)=\cos\theta,$$

since $\boldsymbol{\nu}$ is always turned through an angle $\pi/2$ from $\boldsymbol{t}$ in the positive sense of rotation, which is the sense of circulation $P_hP_{h+1}P_{h+2}$ when we are considering a triangle.

It follows from (3.3), (3.16), and (3.18), that for the point P,

$$\left\{\begin{aligned}
&\left(\frac{dx^1}{ds}\right)_P=\cos\theta, && \left(\frac{dx^2}{ds}\right)_P=\sin\theta,\\
&\left(\frac{d^2x^1}{ds^2}\right)_P=-\gamma\sin\theta, && \left(\frac{d^2x^2}{ds^2}\right)_P=\gamma\cos\theta,\\
&\left(\frac{d^3x^1}{ds^3}\right)_P=(\tfrac{1}{2}K-\gamma^2)\cos\theta-\dot\gamma\sin\theta, &&\\
& && \left(\frac{d^3x^2}{ds^3}\right)_P=(\tfrac{1}{2}K-\gamma^2)\sin\theta+\dot\gamma\cos\theta.
\end{aligned}\right. \tag{3.19}$$

We are finally able to write, as in the preceding section, and *with the same remark concerning the arcs, which are regarded as being small*, the developments of the coördinates x^1, x^2 of a curve extending from the point P in the direction θ measured from the positive direction of the x^1-line, in terms of powers of the arc s; we give these explicitly to the third order inclusive. In the first place, since x^1 and x^2 vanish for $s=0$, we have

$$x^i(s)=s\left\{\left(\frac{dx^i}{ds}\right)_P+\tfrac{1}{2}s\left(\frac{d^2x^i}{ds^2}\right)_P+\tfrac{1}{6}s^2\left(\frac{d^3x^i}{ds^3}\right)_P+③\right\},\qquad i=1,2,$$

where the derivatives at the point P are given by (3.19).

In taking into account the factors $\cos\theta$, $\sin\theta$, we can write

$$\text{(3.20)}\qquad \begin{cases} x^1=\cos\theta\cdot y^1+\sin\theta\cdot z^1,\\ x^2=\cos\theta\cdot y^2+\sin\theta\cdot z^2, \end{cases}$$

where

$$\text{(3.21)}\qquad \begin{cases} y^1=s\left\{1+\tfrac{1}{6}(\tfrac{1}{2}K-\gamma^2)s^2+③\right\},\\ y^2=s\left\{\tfrac{1}{2}\gamma s+\tfrac{1}{6}\dot\gamma s^2+③\right\}; \end{cases}$$

$$\text{(3.22)}\qquad \begin{cases} z^1=s\left\{-\tfrac{1}{2}\gamma s-\tfrac{1}{6}\dot\gamma s^2+③\right\},\\ z^2=s\left\{1+\tfrac{1}{6}(\tfrac{1}{2}K-\gamma^2)s^2+③\right\}. \end{cases}$$

Evidently, the formulas (3.21) are the parametric expressions of the coördinates of a curve C tangent at P to the positive x^1-line, where $\theta=0$, while those of (3.22) refer to a curve tangent to the orthogonal direction x^2.

4. Geodesic curve issuing from P. Angle with the curve at the point P. Length

If we are concerned in particular with a geodesic, so that $\gamma=\dot\gamma=0$, we obtain (3.21) and (3.22) in the form

$$y^1 = s\{1+\tfrac{1}{12}Ks^2+③\},\quad y^2 = s③;$$
$$z^1 = s③,\quad z^2 = s\{1+\tfrac{1}{12}Ks^2+③\}.$$

The parametric representation (3.20) reduces therefore to

$$(4.1)\qquad \begin{cases} x^1 = \cos\theta\cdot s(1+\tfrac{1}{12}Ks^2)+s③, \\ x^2 = \sin\theta\cdot s(1+\tfrac{1}{12}Ks^2)+s③. \end{cases}$$

Consider on C a point P', of which the curvilinear abscissa is l, and suppose that $l>0$; that is, suppose that arcs are considered positive when measured from P toward P'. For simplicity, suppose we have chosen at P the direction of the positive x^1-line to be tangent to C. Then the parametric representation (3.21) applies to C, and we have only to set $s=l$ to obtain the coördinates,

$$l\{1+\tfrac{1}{6}(\tfrac{1}{2}K-\gamma^2)l^2+③\},\quad l\{\tfrac{1}{2}\gamma l+\tfrac{1}{6}\dot\gamma l^2+③\},$$

of the point P'.

The same point P' belongs moreover also to the chord PP', represented parametrically by (4.1) and corresponding to $s=a$. Equating and dividing by l, we obtain

$$(4.2)\qquad \begin{cases} 1+\tfrac{1}{6}(\tfrac{1}{2}K-\gamma^2)l^2+③ = \dfrac{a}{l}\cos\theta\cdot(1+\tfrac{1}{12}Ka^2), \\ \tfrac{1}{2}\gamma l+\tfrac{1}{6}\dot\gamma l^2+③ = \dfrac{a}{l}\sin\theta\cdot(1+\tfrac{1}{12}Ka^2), \end{cases}$$

which define a/l and θ as functions of l and of the quantities γ, $\dot\gamma$ relative to the curve C, and K relative to the surface σ, all evaluated at the point P.

Immediately, dividing the second equation by the first, we obtain

$$\tan\theta = \frac{\tfrac{1}{2}\gamma l+\tfrac{1}{6}\dot\gamma l^2}{1+\tfrac{1}{6}(\tfrac{1}{2}K-\gamma^2)l^2}+③ = \tfrac{1}{2}\gamma l+\tfrac{1}{6}\dot\gamma l^2+③,$$

showing, as was *a priori* evident, that $\tan\theta$ is a small quan-

tity of the first order. Moreover, by the Maclaurin development, we have $\tan\theta=\theta+③$, since $\tan\theta$ is an odd function, and therefore

$$\begin{aligned}\theta&=\tfrac{1}{2}\gamma l+\tfrac{1}{6}\dot{\gamma}l^2+③,\\ \cos\theta&=1-\tfrac{1}{8}\gamma^2l^2+③.\end{aligned}\tag{4.3}$$

With the above value for $\cos\theta$, the first of equations (4.2) gives successively, when we neglect terms of the first order,

$$\frac{a}{l}=1+①,$$

then the second,

$$\frac{a}{l}=1+②,$$

and finally only the third,

$$1+\tfrac{1}{6}(\tfrac{1}{2}K-\gamma^2)l^2=\frac{a}{l}(1-\tfrac{1}{8}\gamma^2l^2)(1+\tfrac{1}{12}Kl^2)+③.$$

In the last binomial we have written l in place of a, since the difference between Kl^2 and Ka^2 is not merely of the third, but actually of the fourth order. We have thus

$$\begin{aligned}\frac{a}{l}&=\{1+\tfrac{1}{6}(\tfrac{1}{2}K-\gamma^2)l^2\}(1+\tfrac{1}{8}\gamma^2l^2)(1-\tfrac{1}{12}Kl^2)+③\\ &=1-\tfrac{1}{24}\gamma^2l^2+③,\end{aligned}$$

or, if we please,

$$a=l\{1-\tfrac{1}{24}\gamma^2l^2+③\},\tag{4.4}$$

a well-known relation between the length of a small arc and its chord, for a plane curve or for a curve of ordinary space.[11] For a curve drawn on an arbitrary surface, where γ denotes its geodesic curvature and a is equal to the length

of the geodesic chord, the result has already been noted by Darboux,[12] who has carried out the calculation through the fourth order inclusive. Even in stopping at the third order we could expect K to be involved, since in the course of the calculation we are concerned with terms of the third order in the parametric development, which depend in general on the first and *second* derivatives of the coefficients of ds^2. The result shows that it is not so; nevertheless, I have not succeeded in accounting for this by any synthetic considerations.

Remark. In formula (4.4), the value of γ corresponds to the point P, one of the two extremities common to the arc and to its chord. To the same order of approximation, we can replace the value of γ at P by its value at the other extremity P', or more generally at an arbitrary point M interior to the arc PP'. Indeed, if $s \leqq l$ is the curvilinear abscissa PM of M, we have, according to the formula of finite increments,

$$\gamma_M = \gamma\{1 + ①\};$$

this assures the validity of (4.4), out to terms of the second order inclusive, for an arbitrary point M of the arc PP' at which we calculate the value of γ.

5. Return to the triangle $P_hP_{h+1}P_{h+2}$. Triangle of geodesic chords. Relations between the sides and the angles of these two triangles

We return to the triangle $P_hP_{h+1}P_{h+2}$ to remark that, from the qualitative point of view with respect to sense and sign, the entire development proceeds, if the triangle is sufficiently small, as in the case of small circular triangles in the plane (§1).

Henceforth, we fix our attention on an arbitrary vertex P_h, and continue to denote by φ_h the angle of the given triangle, formed at P_h by the two curvilinear arcs P_hP_{h+1} and P_hP_{h+2}, by α_h the angle formed at P_h by the corresponding geodesic chords P_hP_{h+1} and P_hP_{h+2}. As in §1, we find it convenient to introduce the two angles β comprised at the point P_h between the arcs P_hP_{h+1}, P_hP_{h+2} and their chords. We designate them $\beta_{h,h+1}$ and $\beta_{h,h+2}$, respectively, *the first index referring to the vertex*. In §1, because of a property of the circle, we had only three angles β, and the first index sufficed. For example, with respect to the arc P_hP_{h+1}, we had two equal angles $\beta_{h,h+1}$ and $\beta_{h+1,h}$, both of which we designated β_{h+2}. Here we have introduced six β's with two distinct indices each. Their evaluation has been substantially made in the preceding section: we have only duly to specify formula (4.3).

We apply formula (4.3) first to the angle $\beta_{h,h+1}$ comprised at the point P_h between the arc P_hP_{h+1} and its chord. Then $\beta_{h,h+1}$ identifies itself with θ provided we replace l by the length l_{h+2} of the arc P_hP_{h+1} opposite the vertex P_{h+2}, and γ, $\dot{\gamma}$ by γ_{h+2}, $\dot{\gamma}_{h+2}$, or more specifically by $(\gamma_{h+2})_{P_h}$, $(\dot{\gamma}_{h+2})_{P_h}$, if we wish to exhibit the point of the arc P_hP_{h+1} to which the value of the curvature and its derivative refer. Thus we have

$$\beta_{h,h+1}=\tfrac{1}{2}(\gamma_{h+2})_{P_h}l_{h+2}+\tfrac{1}{6}(\dot{\gamma}_{h+2})_{P_h}l_{h+2}^2+③. \tag{5.1}$$

The sign of $\beta_{h,h+1}$ is that of the preponderant term; that is to say, since the lengths l are essentially positive, the sign of $(\gamma_{h+2})_{P_h}$. This follows necessarily in conformity with remarks made in §1 relative to formula (1.1). Of course, we can give to h the values 1, 2, 3, with our usual convention of reducing the indices $h+1$, $h+2$ modulo 3. The notation $\cdot$ indicates differentiation with respect to the length of the

arc P_hP_{h+1}, taken from P_h toward P_{h+1}, that is, in *the sense $P_hP_{h+1}P_{h+2}$ adopted as positive for tracing the perimeter of the triangle.*

Consider now the other arc P_hP_{h+2} abutting on the point P_h, and the angle $\beta_{h,h+2}$ which it forms with the corresponding chord P_hP_{h+2}. Its expression, furnished directly by (4.3), is

$$\beta_{h,h+2}=\tfrac{1}{2}(\gamma_{h+1})_{P_h}l_{h+1}+\tfrac{1}{6}(\dot{\gamma}_{h+1})l_{h+1}^2+③,$$

where the index $h+1$ refers to the arc P_hP_{h+2}, opposite the vertex P_{h+1}: here evidently we must regard the derivative $\dot{\gamma}_{h+1}$ of the curvature as being evaluated, with respect to the arc P_hP_{h+2}, for the direction toward P_{h+2}. The sense does not coincide this time with the positive sense of circulation $P_hP_{h+1}P_{h+2}$, but is reversed. If *we make the convention that the superposed dot $\cdot$ always designates a derivative evaluated in the sense of circulation $P_hP_{h+1}P_{h+2}$, adopted as positive on the perimeter of the triangle*, we must replace the sign $+$ by the sign $-$ in the second member of the above formula. We then obtain

$$\beta_{h,h+2}=\tfrac{1}{2}(\gamma_{h+1})_{P_h}l_{h+1}-\tfrac{1}{6}(\dot{\gamma}_{h+1})_{P_h}l_{h+1}^2+③. \tag{5.2}$$

The rule to remember is that in the expressions of the β's of two indices, if the order of the indices is progressive, as in formula (5.1), the second term has the sign $+$, while if the order of the indices is retrogressive, as in formula (5.2), the second term has the sign $-$.

In formulæ (5.1) and (5.2) appear γ and $\dot{\gamma}$ relative to one same arc, namely, for a convenient choice of indices, $P_{h+1}P_{h+2}$, related one time to P_{h+1}, the other time to P_{h+2}. To verify that this is so, we write $h+1$ in place of h in for-

mula (5.1), and take account of the point at which the evaluation is made, obtaining

$$\beta_{h+1,h+2} = \tfrac{1}{2}(\gamma_h)_{P_{h+1}} l_h + \tfrac{1}{6}(\dot{\gamma}_h)_{P_{h+1}} l_h^2 + ③;$$

and $h+2$ in place of h in (5.2), obtaining

$$\beta_{h+2,h+1} = \tfrac{1}{2}(\gamma_h)_{P_{h+2}} l_h - \tfrac{1}{6}(\dot{\gamma}_h)_{P_{h+2}} l_h^2 + ③.$$

We find it advantageous to make our evaluations, for these two formulæ, at the midpoint M_h of the arc $P_{h+1}P_{h+2}$. We have, for the positive sense of circulation,

$$(\gamma_h)_{M_h} l_h = (\gamma_h)_{P_{h+1}} l_h + \tfrac{1}{2}(\dot{\gamma}_h)_{P_{h+1}} l_h^2 + ③,$$
$$(\gamma_h)_{P_{h+2}} l_h = (\gamma_h)_{M_h} l_h + \tfrac{1}{2}(\dot{\gamma}_h)_{M_h} l_h^2 + ③.$$

Noting that

$$(\dot{\gamma}_h)_{P_{h+1}} l_h^2 = (\dot{\gamma}_h)_{M_h} l_h^2 + ③ = \dot{\gamma}_h l_h^2 + ③,$$

where we have suppressed the specification as to point when the evaluation is made at M_h, we obtain

$$(\gamma_h)_{P_{h+1}} l_h = \gamma_h l_h - \tfrac{1}{2}\dot{\gamma}_h l_h^2 + ③,$$
$$(\gamma_h)_{P_{h+2}} l_h = \gamma_h l_h + \tfrac{1}{2}\dot{\gamma}_h l_h^2 + ③,$$

and the preceding expressions for $\beta_{h+1,h+2}$ and $\beta_{h+2,h+1}$ take the definitive forms

$$\beta_{h+1,h+2} = \tfrac{1}{2}\gamma_h l_h - \tfrac{1}{12}\dot{\gamma}_h l_h^2 + ③, \tag{5.3}$$
$$\beta_{h+2,h+1} = \tfrac{1}{2}\gamma_h l_h + \tfrac{1}{12}\dot{\gamma}_h l_h^2 + ③, \tag{5.4}$$

where it is always the first index which denotes the vertex to which the angle is related; and γ_h and $\dot{\gamma}_h$ denote their determination at the midpoint M_h of the arc $P_{h+1}P_{h+2}$.

There are formulæ analogous to (1.2) expressing the angle φ_h of the given triangle by means of the corresponding angle

α_h of the triangle of chords and by means of the β's. They are precisely

$$\varphi_h = \alpha_h + \beta_{h,h+1} + \beta_{h,h+2};$$

when we solve for α_h, we have, according to (5.3) and (5.4),

$$\alpha_h = \varphi_h - \tfrac{1}{2}(\gamma_{h+1} l_{h+1} + \gamma_{h+2} l_{h+2}) - \tfrac{1}{12}(\dot{\gamma}_{h+1} l_{h+1}^2 - \dot{\gamma}_{h+2} l_{h+2}^2) + ③, \qquad h = 1, 2, 3. \tag{5.5}$$

In the relations (5.5), as well as in the relations (4.3) from which they were derived, the total curvature K of the surface σ in the neighborhood of the triangle does not figure in terms up to the second order inclusive. The same is true of the expressions of the lengths of the geodesic chords as functions of the curvilinear sides. Indeed, formula (4.4), related to the arc $P_{h+1}P_{h+2}$ of length l_h, gives, for the geodesic chord a_h,

$$a_h = l_h\{(1 - \tfrac{1}{24}\gamma_h^2 l_h^2) + ③\}, \tag{5.6}$$

where it is permissible, according to the last remark in §4, to regard γ_h as being determined at the midpoint M_h of the arc $P_{h+1}P_{h+2}$.

We have in formulæ (5.5), (5.6) all that we shall need in the next section in order to establish, to second approximations, the trigonometry of small curvilinear triangles on σ.

6. TRIGONOMETRIC RELATIONS

The triangle of geodesic chords has sides a_h and angles α_h. In our approximation, *disregarding terms of the third and higher orders*, we can consider the surface σ, in the neighborhood of an arbitrary point O of σ, as *a variety of constant curvature*, K, the determination of K being computed at the point O.[13]

It follows that, for the geodesic triangle having sides a_h

and angles α_h, the formulæ of spherical or Lobatschewskian trigonometry are valid, depending on the sign of K. We set

$$K=\frac{1}{R^2},$$

with

$$R>0 \text{ for } K>0,$$

and

$$-iR>0 \text{ for } K<0.$$

Confining ourselves to the theorem of sines, we must transform, either for the sphere or for the pseudosphere, the formulæ

$$|\sin a_h'|=\frac{l}{|R|}\sin\alpha_h, \qquad h=1, 2, 3, \tag{6.1}$$

where for simplicity we have put

$$a_h'=\frac{a_h}{R}.^{14} \tag{6.2}$$

Now l, independent of h, is called the *modulus* of the spherical or pseudospherical triangle. Its value, expressed for example by means of the sides alone, is given by

$$\frac{l}{|R|}=\frac{|\sin a_1' \sin a_2' \sin a_3'|}{\{1-\cos^2 a_1'-\cos^2 a_2'-\cos^2 a_3'+2\cos a_1'\cos a_2'\cos a_3'\}^{\frac{1}{2}}}. \tag{6.3}$$

Some interesting remarks might be made about the expression (6.3) of l, but for the sake of brevity we shall not dwell on them. We restrict ourselves here again to developing the *theorem of sines;* that is to say, we introduce in formulæ (6.1) the expressions (5.5) of the α_h, and the expressions (6.2), (5.6) of the a_h, which may be written

$$a_h=\frac{l_h}{R}(1-\tfrac{1}{24}\gamma_h^2 l_h^2+\textcircled{3}). \tag{6.4}$$

This is just what we need for our specified order of approximation. We have only to solve finally for the side l_h of the curvilinear triangle. One such calculation already has been made, to the first approximation, in §1; it led us to the expressions (1.10) of the l_h as functions of the angles φ_h (and of the γ_h and the auxiliary l). In this connection, it is important to remark that the auxiliary length l, introduced actually as factor of proportionality in formulæ (6.1), is equal in the first approximation to the quantity designated by the same letter in §1.

Indeed, starting from equations (6.1) and taking account of the fact that a_h' is either real or pure imaginary, so that $a_h'^2$ is real in any case, we obtain immediately

$$|a_h'|\{1-\tfrac{1}{6}a_h'^2+④\} = \frac{l}{|R|}\sin\alpha_h.$$

Replacing $|a'|$ by its value (6.4), and noting that in virtue of formulæ (6.4),

$$a_h' = \frac{l_h}{R}\{1+②\},\quad 1-\tfrac{1}{6}a_h'^2 = 1-\frac{1}{6}\frac{l_h^2}{R^2}+④,$$

we obtain

$$\frac{l_h}{|R|} = \frac{l}{|R|}\sin\alpha_h\left\{1+\frac{1}{6}\frac{l^2}{R^2}\sin^2\alpha_h+\tfrac{1}{24}\gamma_h^2\, l_h^2+③\right\},$$

whence in particular

$$l_h = l\sin\alpha_h\{1+②\},$$

which coincides, up to terms of the first order, with the characterization of l which follows from (1.5) by means of (1.6).

In the parenthesis in the above formula for $l_h/|R|$, when we take account of the second order also, we can simply replace α_h by φ_h and likewise l_h by $l\sin\varphi_h$, since in doing

so we neglect only the third order relative to unity. We have therefore, writing K in place of $1/R^2$,

$$l_h = l \sin \alpha_h \{1 + \tfrac{1}{6} l^2 \sin^2 \varphi_h (K + \tfrac{1}{4}\gamma_h^2) + ③\}. \tag{6.5}$$

There is left only the evaluation of $\sin \alpha_h$, up to the second order inclusive, as function of the φ_h and l.

Writing for the moment

$$\delta_h = \tfrac{1}{2}(\gamma_{h+1} l_{h+1} + \gamma_{h+2} l_{h+2}) + \tfrac{1}{12}(\dot\gamma_{h+1} l_{h+1}^2 + \dot\gamma_{h+2} l_{h+2}^2), \tag{6.6}$$

we obtain (5.5) in the form

$$\alpha_h = \varphi_h - \delta_h + ③,$$

whence

$$\sin \alpha_h = \sin \varphi_h \cos \delta_h - \cos \varphi_h \sin \delta_h + ③.$$

Taking account of the circumstance that δ_h is of the first order, and neglecting as always the third order relative to finite terms, we obtain

$$\sin \alpha_h = \sin \varphi_h (1 - \tfrac{1}{2}\delta_h^2) - \cos \varphi_h \cdot \delta_h + ③.$$

In δ_h^2, according to (6.6), we may retain only the square of the first term. There remains therefore

$$\begin{aligned} \sin \alpha_h = \sin \varphi_h \{1 - \tfrac{1}{8}(\gamma_{h+1} l_{h+1} + \gamma_{h+2} l_{h+2})^2\} \\ - \cos \varphi_h \cdot \delta_h + ③, \end{aligned} \tag{6.7}$$

where δ_h is defined by (6.6). The terms of orders 0 and 1 already have been calculated in §1, formula (1.8). Accordingly, they can be written, by (1.9), $\sin \varphi_h$ and $\psi_h \sin \varphi_h$, where

$$\psi_h = \tfrac{1}{2} l (\gamma_h \cos \varphi_h - 3\tau \cot \varphi_h). \tag{6.8}$$

On the other hand, the term $\chi_h \sin \varphi_h$ of second order in (6.7) is, by (6.6),

$$-\sin\varphi_h \cdot \tfrac{1}{8}(\gamma_{h+1}l_{h+1}+\gamma_{h+2}l_{h+2})^2$$
$$-\cos\varphi_h \cdot \tfrac{1}{12}(\dot{\gamma}_{h+1}l_{h+1}^2+\dot{\gamma}_{h+2}l_{h+2}^2).$$

Here we may replace l_{h+1} and l_{h+2} by $l\sin\varphi_{h+1}$ and $l\sin\varphi_{h+2}$, and obtain

$$\chi_h = -l^2\{\tfrac{1}{8}(\gamma_{h+1}\sin\varphi_{h+1}+\gamma_{h+2}\sin\varphi_{h+2})^2 + \tfrac{1}{12}\cot\varphi_h(\dot{\gamma}_{h+1}\sin^2\varphi_{h+1}+\dot{\gamma}_{h+2}\sin^2\varphi_{h+2})\}.$$

For symmetry, we introduce as in §1 the triangular curvature

$$\tau = \tfrac{1}{3}\sum_1^3{}_h\,\gamma_h\sin\varphi_h, \tag{6.9}$$

and the analogous trinomial

$$\tau' = \tfrac{1}{3}\sum_1^3{}_h\,\dot{\gamma}_h\sin^2\varphi_h. \tag{6.10}$$

Now we can give to the expression of χ_h the form

$$\chi_h = -l^2\{\tfrac{1}{8}(3\tau-\gamma_h\sin\varphi_h)^2 + \tfrac{1}{12}\cot\varphi_h(3\tau'-\dot{\gamma}_h\sin^2\varphi_h)\}. \tag{6.11}$$

We always can write (6.7), when we separate terms of different orders, in the form

$$\sin\alpha_h = \sin\varphi_h \cdot \{1+\psi_h+\chi_h+③\}. \tag{6.12}$$

Noting that $\sin\alpha_h$ may be replaced by $\sin\varphi_h$ when it is multiplied by a term of the second order, we obtain finally from (6.5) and (6.12)

$$\frac{l_h}{|R|\sin\varphi_h} = l\{1+\psi_h+\chi_h+\tfrac{1}{6}l^2\sin^2\varphi_h\cdot(K+\tfrac{1}{4}\gamma_h^2)+③\}. \tag{6.13}$$

This is the form of the theorem of sines to the second approximation, with ψ_h and χ_h having values (6.8) and (6.11).

7. REMARK ON THE THEOREM OF LEGENDRE

In the calculations of the preceding section we have used, for the triangle of chords, the formulæ of spherical or pseudospherical trigonometry. One might perhaps be inclined to think that, in the approximation of the second order with which we have contented ourselves, one could apply first the theorem of Legendre, which, to the same degree of approximation, reduces the resolution of a geodesic triangle on a surface of constant curvature to that of a plane triangle having the same sides, and angles modified by one-third of the excess $(\alpha_1+\alpha_2+\alpha_3)-\pi$. Actually, this is not the case; and here is the reason.

One arrives at the above enunciation of Legendre's theorem by establishing the fact that the same three independent relations are satisfied up to the second order inclusive by the elements of a spherical or pseudospherical triangle and those of the corresponding plane triangle. It does not follow in an absolute manner, since we are not dealing with rigorous equalities, that every combination of three relations is equally valid to the same order of approximation. Precisely in the theorem of sines we see that the equivalence subsists only in the first, but not in the second approximation.

That is why we have had to take as point of departure the formulæ (6.1) of spherical or pseudospherical trigonometry, without first reducing them by means of the theorem of Legendre.

8. GENERAL INDICATIONS CONCERNING APPROXIMATIONS OF HIGHER ORDER

I. **The plane case.** The parametric representation of plane curves, considered in §2, can be extended, with the

aid of the Taylor development according to increasing powers of the arc s, up to an arbitrary order m. We make the hypothesis, of course, that each arc of curve in question behaves, with respect to the geodesic curvature γ and its derivatives, in a manner analogous to that admitted for the first two approximations. We suppose in particular that if l is the arc length and

$$\gamma, \dot{\gamma}, \cdots, \gamma^{(m-1)}, \gamma^{(m)}$$

the values of the curvature and its derivatives at an arbitrary point of the arc, then the products

$$l\gamma, \ l^2\dot{\gamma}, \ \cdots, \ l^m \gamma^{(m-1)}, \ l^{m+1}\gamma^{(m)},$$

which are pure numbers, are of order not less than

$$1, 2, \cdots, m, m+1,$$

respectively. Then, if we consider first the unit vector t, tangent to the curve, we see, from the formulæ of Frenet, that $t(s)$ can be represented, up to the order m inclusive, in the form

$$t(s) = A(s)t_0 + B(s)n_0. \tag{8.1}$$

Here $A(s)$ and $B(s)$ are polynomials of degree m in s, the coefficients being polynomials in the values of γ and its derivatives, $\dot{\gamma}, \cdots, \gamma^{(m-1)}$, evaluated for $s=0$, or just as well for any other value of s between 0 and l; t_0 designates the value of t for $s=0$, and n_0 is t_0 turned through an angle $\pi/2$ in the positive sense.

If $Q(s)$ designates the variable point of the arc under consideration, we have

$$\frac{dQ}{ds} = t(s),$$

so that, by integration of $t(s)$ from $s=0$, corresponding,

say, to the point P, up to a generic s, we obtain the parametric representation of the arc. For brevity we put

$$\mathscr{C}(s) = \int_0^s \mathscr{A}(s)ds, \quad \mathscr{D}(s) = \int_0^s \mathscr{B}(s)ds,$$

the γ's figuring in the preceding A and B being treated as constants relative to the integration, and obtain for the other extremity P' of the arc, corresponding to $s=l$, $Q=P'$,

$$P'-P=\mathscr{C}(l)\boldsymbol{t}_0+\mathscr{D}(l)\boldsymbol{n}_0, \tag{8.2}$$

while, for the tangential versor (unit vector) $\boldsymbol{t}'$ at the point P', we set $s=l$ in (8.1) and obtain

$$\boldsymbol{t}'=A(l)\boldsymbol{t}_0+B(l)\boldsymbol{n}_0. \tag{8.3}$$

On the other hand, if $\boldsymbol{u}$ designates the versor of $P'-P$ and a its length, we have

$$P'-P=a\boldsymbol{u},$$

so that we can give to the vectorial equation (8.2) the form

$$a\boldsymbol{u}=\mathscr{C}(l)\boldsymbol{t}_0+\mathscr{D}(l)\boldsymbol{n}_0. \tag{8.4}$$

Now (8.4) is equivalent to two scalar equations, while (8.3), expressing the equality of two versors, yields only one. The set of three equations allows us to deduce, for the lunar figure formed by an arc PP' and its chord PP', both the length a of the chord and also the values of the two angles between chord and arc at vertices P, P', as functions of $l, \gamma, \dot{\gamma}, \cdots, \gamma^{(m-1)}$.

If now we consider, in the plane, a small triangle $P_hP_{h+1}P_{h+2}$ having arbitrary curves as sides, we accordingly can obtain for each side the length a_h of the chord and the two angles $\beta_{h+1,h+2}$, $\beta_{h+2,h+1}$ as functions of $l_h, \gamma_h, \dot{\gamma}_h, \cdots, \gamma_h^{(m-1)}$.

And we have, as in §4, for the angles α_h of the triangle of chords,

$$\alpha_h = \varphi_h - (\beta_{h+1,h+2} + \beta_{h+2,h+1}),$$

the φ_h being the angles of the curvilinear triangle.

Thus we have only to replace, in an arbitrary relation of rectilinear trigonometry, the a_h and the α_h as functions of $l_h, \varphi_h, \gamma_h, \dot{\gamma}_h, \cdots, \gamma_h^{(m-1)}$, in order to deduce the corresponding formula, relative to the curvilinear triangle, *exactly up to the order m inclusive.*

II. **The sphere.** All the preceding can be carried over, *mutatis mutandis*, to arbitrary small curvilinear triangles on a sphere or pseudosphere, proper account being taken, in the course of the calculations and in the final formulæ, of the value K of the constant total curvature.

III. **Arbitrary surface.** The method followed in §§3–5, with $m=2$, for an arbitrary surface, and the preceding supplementary sketch, in the case of the plane, for an arbitrary m, can be combined. *We accordingly introduce*, not only the derivatives of the geodesic curvature up to the order $m-1$, but also *the values of the total curvature K and of its partial derivatives up to the order* $m-2$, these last evaluated at an arbitrary point of the triangle.

It will be helpful, in connection with our general considerations, to recall a circumstance which is well known for geodesic triangles.[15] It is that we generally cannot attain relations involving *only* sides, angles, geodesic curvature γ of the sides, total curvature K of the surface, and their derivatives. This is because of the metric non-homogeneity of the surface σ, which makes the position and orientation of the triangle significant in the computations. In general, therefore, we cannot eliminate the coördinates of the vertices, or any other absolute element which we might

introduce in order to effect the calculations or to fix the values of K and its derivatives at a certain point. We are assured *a priori* that this difficulty does not exist for surfaces admitting a three-parameter group of motions; but such surfaces are surfaces of constant total curvature, which we already have treated.

IV. **First and second approximations.** We have treated in detail the cases $m=1$ and $m=2$. It is true that for $m=1$ we have first relied (§1) on elementary geometry in the plane; but we have later pointed out, at the end of §5, that the results for the first approximation are independent of the curvature K, since K enters only in terms starting with the second. *In the first approximation, all our results concerning small plane triangles formed by three circular arcs hold in the general case.*

For $m=2$, the analysis developed in §§4–6 reveals that the curvature K, which figures only in terms of the second order, can be used as evaluated at a single point of the triangle. There is therefore no difference between the general case and the case $K=$constant, and we may if we wish consider that we are dealing with the sphere if $K>0$, or with the plane if $K=0$, or with the pseudosphere if $K<0$.

It is perhaps of some interest to add one more word concerning the canonical model which proves suitable for the sides of our curvilinear triangle.

For $m=1$, as we have just remarked, each side can be identified with a small circular arc, an arc of its osculating circle.

For $m=2$, we must take account also of γ for those curves on which the geodesic curvature is not constant; otherwise the choice of canonical type is arbitrary. Thus, confining ourselves specifically to the sphere, we can for example take recourse to loxodromic curves; and it is perfectly permis-

sible for us to restrict ourselves, for $m=2$, to spherical triangles formed by loxodromic arcs: we must however use loxodromic arcs related in general to different poles for the three sides.

NOTES

[1]See in particular Darboux, *Leçons sur la théorie générale des surfaces*, t. 3 (Paris, Gauthiers-Villars, 1894), Livre VI, Chap. VIII.

[2]"Sulla curvatura delle superficie e variatà," *Rend. del Circolo Mat. di Palermo*, t. XLII (1917), pp. 227–259.

[3]"Terne di congruenze sopra una superficie ed estensione delle trigonometria," *Compositio Mathematica*, vol. 1 (1934), pp. 115–162. See also A. Tonolo, "Il teorema dei seni per i triangoli infinitesimi tracciati sopra una superficie," ibid., vol. 2 (1935), pp. 424–437.

[4]"Der approximative Sinussatz für kleine Dreiecke auf krummen Flächen," ibid., vol. 3 (1936), pp. 52–54.

[5]In the relations (10′) and (14), given in Chapter III of my cited memoir, there is apparently a change of sign with regard to the binomial $\gamma_h\cos\varphi_h - 3\tau\cot\varphi_h$. This comes from the circumstance that the angles designated there by ψ_h do not coincide with the actual φ_h, but with their supplements $\pi-\varphi_h$, which are angles exterior to the triangle.

[6]This is exactly the same as the convention adopted in the preceding section relative to the curvature of the sides of a curvilinear triangle; the sense of the couple $\boldsymbol{t}$, $\boldsymbol{n}$ is fixed, in the case of the triangle, as follows: $\boldsymbol{t}$, for each side, is directed in the sense of describing $P_h\,P_{h+1}\,P_{h+2}$; $\boldsymbol{n}$ is directed toward the interior of the triangle.

[7]This deduction from the formulæ of Frenet, and also the deduction for curves in space, may be found in all the modern treatises on differential geometry. See for instance W. C. Graustein, *Differential Geometry*, New York, Macmillan, 1935, p. 39; or W. Blaschke, *Vorlesungen über Differentialgeometrie*, Berlin, Springer, B. I (3. Auflage, 1930, p. 26); or G. Julia, *Eléments de Géométrie infinitésimale*, Paris, Gauthier-Villars, 2 ed., 1936, p. 129.

[8]See for instance my *Absolute Differential Calculus*, edited by Dr. E. Persico, Glasgow and London, Blackie, 1927, pp. 139–140.

[9]We could dispense with the direct verification by referring to the general case of a Riemannian variety of an arbitrary number n of dimensions. The extension of the formulæ of Frenet has been pointed out by Blaschke, "Frenet's Formeln für den Raum von Riemann," *Math. Zeitschrift*, B. 6 (1920), pp. 94–99, and subsequently has been the object of a large number of publications. See in particular L. P. Eisenhart, *Riemannian Geometry*, Princeton University Press, 1927, pp. 103–107.

[10]See my note, "Forme canoniche dei ds^2 binari con data curvatura totale," *Rend. Acc. Lincei*, vol. 25 (1937), pp. 197–205.

[11]It is obtained immediately from the canonical representation of a curve in the neighborhood of one of its points. See for example Julia, *loc. cit.*, note 7, p. 132. E. Cesaro has obtained it by direct considerations on page 194 of his *Elementi di calcolo infinitesimale*, Napoli, Alvano, 1905.

[12]*Loc. cit.*, note 1, p. 176.

[13]The remark is almost trivial since, in the canonical representation of a curve in the neighborhood of a given point of the curve, the variation of the curvature affects only the terms in the fourth and higher powers of s. For more details one can see my recent note, previously cited, note 10, as well as the paper of Professor Severi, *loc. cit.*, note 2, especially pp. 232–241.

[14]In the case of the sphere, with $R>0$, the first members of equations (6.1) have to be $\sin(a_h/R)$, that is to say $\sin a_h'$, which is equal to $|\sin a_h'|$, for a_h' positive and sufficiently small. In the non-Euclidean case, where R is a pure imaginary with $-iR>0$, we have only to replace $\sin(a_h/R)$ by S, the hyperbolic sine, of the argument $a_h/|R|$. To show that it is again $|\sin a_h'|$, we start from

$$a_h' = a_h/R = -ia_h/|R|,$$

whence

$$\sin a_h' = -\sin ia_h/|R| = -iS(a_h/|R|).$$

Equating the moduli of the two members, we obtain

$$|\sin a_h'| = S(a_h/|R|).$$

See for example Bianchi, *Lezioni di geometria differenziale*, t. I (3rd ed., Pisa, 1922, p. 638).

[15]Cf. Darboux, *loc. cit.*, note 1, pp. 181–182.

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